Average and Instantaneous Speeds: Introducing Limits

There are two types of **speeds**: (1) **average speed over a time interval** and (2) **instantaneous speed** (speedometer reading at an instant in time.)

If you are travelling at a constant speed, then the two are the same! However, if you are not travelling at varying speeds, then they are different. For example, you might average 70 miles per hour travelling from Murfreesboro to Nashville, but even if you are using cruise control your speedometer reading will change as you pass a car or slow down to avoid rear-ending someone. The speedometer reading at an instant of time is the instantaneous speed at that time.

How do you compute average speed over a time interval? It is total distance travelled during that time interval divided by the length of the time interval (the elapsed time.) For example, if you travelled 40 miles from Murfreesboro to Nashville in 30 minutes, then your average speed would be $\frac{40\ miles}{\frac{1}{2}\,hour}$ or $80\ \frac{miles}{hour}$.

Suppose you are travelling along a straight flat road in the direction that the mile markers are getting bigger. You can think of this as driving on a number line. The average speed in the time interval $[a, b]$ is $\dfrac{s(b)-s(a)}{b-a}$ where $s(t)$ is the position of the car along the road (number line) at time t. (Make sure you always realize that s stands for position and not speed). $s(b)$ is the position of the car along the number line at time $t = b$ and $s(a)$ is the position at time $t = a$, so $s(b) - s(a)$ is the change in position or distance travelled. The length of the time interval (the elapsed time) is $b - a$. So, in this case where the motion is such that the mile markers are getting bigger, $\dfrac{s(b)-s(a)}{b-a}$ is just the total distance travelled divided by elapsed time.

Suppose you are in a car on a straight flat road. The car accelerates from 0 to 55 miles per hour in 10 seconds, with a constant rate of acceleration. Then after 10 seconds, the car travels at a constant speed of 55 miles per hour.

The position function for the car is given by

$$s(t) = \begin{cases} \left(\frac{121}{30}\right)t^2 & for\ 0 \le t \le 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & for\ t > 10 \end{cases}$$

where t is measured in seconds and position s is measured in feet.

EXERCISE 1: Place the car in its proper position along the road at the following times:

$$t = 0, t = 2, t = 4, t = 6, t = 8, t = 10, t = 12, t = 14, t = 16, t = 18$$

That is, evaluate $s(0)$, $s(2)$, $s(4)$, $s(6)$, $s(8)$, $s(10)$, $s(12)$, $s(14)$, $s(16)$, $s(18)$ and mark the location at the requested times on a number line.

EXERCISE 2: Calculate the average speed of the car during each of the time intervals in the table below. Set your calculator to floating point mode, and write all the decimal digits that appear in your calculator display or use www.wolframalpha.com and truncate after 8 digits to the right of the decimal point. (Do not write down intermediate answers and re-enter these in your calculator; instead let the calculator evaluate the entire expression for average speed to avoid roundoff error propagation and calculator entry error.)

Time Interval	Average speed in the time interval (in ft. per second)	Time Interval	Average speed in the time interval (in ft per second)
[1, 1.0001]		[.9999,1]	
[1, 1.00001]		[.99999,1]	
[1, 1.000001]		[.999999,1]	
[1, 1.0000001]		[.9999999,1]	

EXERCISE 3: Based on the average speed calculations in Exercise 2, predict the instantaneous speed of the car at t = 1 second. Give your answer in ft per second and then convert from ft/sec to mi/hr. (There are 5280 ft in a mile.) You have predicted the speedometer reading of the car at t = 1 second. Is this reasonable?

EXERCISE 4: Create a table similar to the one in Exercise 2 and use the table to predict the instantaneous speed of the car at t = 8 seconds. Show the table below.

EXERCISE 5: It turns out that the following is the speed function for the motion in this problem. (Note that the speed and velocity are the same in this problem, since the motion is such that the mile (or foot) markers are getting larger, that is the motion is in the positive direction. When motion is in the positive direction, velocity is positive, and speed and velocity are the same. When the motion is in the direction in which the mile (or foot) markers get smaller, we say the motion is in the negative direction, and the velocity is the negative of the speed.)

$$v(t) = \begin{cases} \left(\frac{121}{15}\right)t & for\ 0 \leq t \leq 10 \\ \frac{242}{3} & for\ t > 10 \end{cases}$$ where t is in ft/sec and t is in seconds.

Compute v(1) and v(8) using the function above. Does this agree with your previous predictions?

Teaching Notes

1) This lesson is typically the second lesson I teach during calculus 1. To familiarize students with what is meant by a position function s(t) for motion along a line, the first lesson in the semester typically involves a whole group class activity involving students walking along a number line in front of a motion detector (Texas Instruments (TI) Calculator Based Ranger (CBR)) to produce the position function for their walk and then analyzing how the walk was related to the graph of the position function. Students must understand that s(t) is not the speed at time t and that if motion is in the direction of increasing numbers on the number line, that $s(b) - s(a)$ is just the distance travelled during the time interval $[a, b]$.

2) I set this lesson up by telling students that calculus is a method of calculation based on the concept of the limit and we are going to start our study of calculus this semester by seeing some examples of limits. These examples will help us to understand the concept of a limit and why it is important and then a bit later we will develop more efficient ways of computing limits. Now we are exploring the concept.

3) I typically do Exercises 1,2, and 3 as a whole class activity and then assign the rest for homework. In a discussion of exercises 2 and 3, I ask questions like the following for columns 1 and 2 and then also for columns 3 and 4.

    - *What is happening to the time intervals in the table in exercise 2 as we go down the table?* (I want them to notice that the time intervals are getting shorter and are "shrinking down upon the instant in time $t = 1$.")
    - *Is there a pattern in your results for the average speeds you computed as you go down the table?* (I want them to notice that extra 6's appear in each new table entry.)
    - *Are the average speeds approaching a particular number as we go further and further down in the table?* (I would like for them to answer $8.0\overline{6}\,\frac{ft}{s}$)
    - *Can you make the average speeds as close as you please to* $8.0\overline{6}\,\frac{ft}{s}$ *? How?* (I want them to realize that by going sufficiently far down in the table, in other words, by making the time intervals sufficiently small, the average speeds can be made as close to $8.0\overline{6}\,\frac{ft}{s}$ as one might demand.
    - Suppose *someone wants the average speed to be within .0000001 of* $8.0\overline{6}\,\frac{ft}{s}$. *Is it possible to get a result in the table that close to* $8.0\overline{6}\,\frac{ft}{s}$*? How?"*

    Although I do not look at the informal definition of limit on this day, I do introduce the term "limit" as this point in time, often saying that we have found that the average speeds have a limiting value.

    These questions foreshadow the informal definition of limit which we will introduce in the text on a subsequent day; when we introduce this definition, we refer back to this activity. However, this can lead to a point of confusion that must be addressed. In this activity we are predicting the limit of a difference quotient $\frac{s(t)-s(1)}{t-1}$ instead of the limit of $s(t)$. The informal definition of limit in texts is stated in terms of $f(x)$ or in this case $s(t)$. I like to use a sticky note in place of the $f(x)$ when presenting the informal definition and point out that the function we are taking the limit of could be a combination of functions like a difference quotient or just a single function.

I also ask "Are the average speeds in column 2 approaching the same value as the average speeds in column 4?" This provides context for a future discussion of right- and left-hand limits and the fact that for the limit to exist the right- and left-hand limits must be the same.

4) I find it helpful to call the problem presented in this activity, "The Broken Speedometer Problem."

   You need to know the speedometer reading at an instant of time, but the speedometer is broken, and you cannot just look down at the speedometer at that instant of time. But you do know the position function s(t). How can you use the position function s(t) to determine the speedometer reading at a specific instant in time?

   I think this helps students think about the process that is being introduced in this activity. The day after we do the whole group activity with exercises 1, 2, and 3 and assign exercises 4 and 5 for homework, I ask students to explain the process that we are using to solve the broken speedometer problem. I am wanting them to talk about examining average speeds in time intervals shrinking down upon a particular instant in time and observing a limiting value for the average speeds. I ask if it seems reasonable that the speedometer reading at $t = 1$ for example is very close to the average speed in a very short time interval around $t = 1$?

5) This activity is an opportunity to help with formative assessment at the beginning of the semester. Common mistakes are writing time intervals $[a, b]$ such that a is not less than b; lack of parenthesis when entering expressions of the form $\dfrac{s(b)-s(a)}{b-a}$ in the calculator, and instead of changing the intervals from $[1, 1.001]$ etc to $[8, 8.001]$ going from exercise 2 to exercise 4, using $[1, 8.001]$ etc. As I am asking leading questions regarding exercises 2 and 3, I point out that the time interval [a, b] means a ≤ t ≤ b and ask *"what is the difference in the time intervals in columns 1 and 3?"* Some students will not understand the interval notation.

6) A next lesson after this one typically is introducing the tangent problem with the function

$$f(x) = \begin{cases} \left(\frac{121}{30}\right)x^2 & for\ 0 \le x \le 10 \\ \left(\frac{1}{3}\right)(242x - 1210) & for\ x > 10 \end{cases}$$

. The goal is to give students another context in which limits are useful and for students to discover the equivalence of the tangent problem to the broken speedometer problem. I want students to discover that the computation of the slope of the secant line connecting the points $(1, f(1))$ and $(1.01, f(1.01))$ is exactly the same as the computation of the average speed in the time interval $[1, 1.001]$, so the slope of the tangent line at *x = 1* will be the same as the instantaneous velocity at $t = 1$. I have students do the secant line slope computations in an activity similar to this one, and after doing a few of the computations ask if this looks familiar in case a student has not already pointed this out.