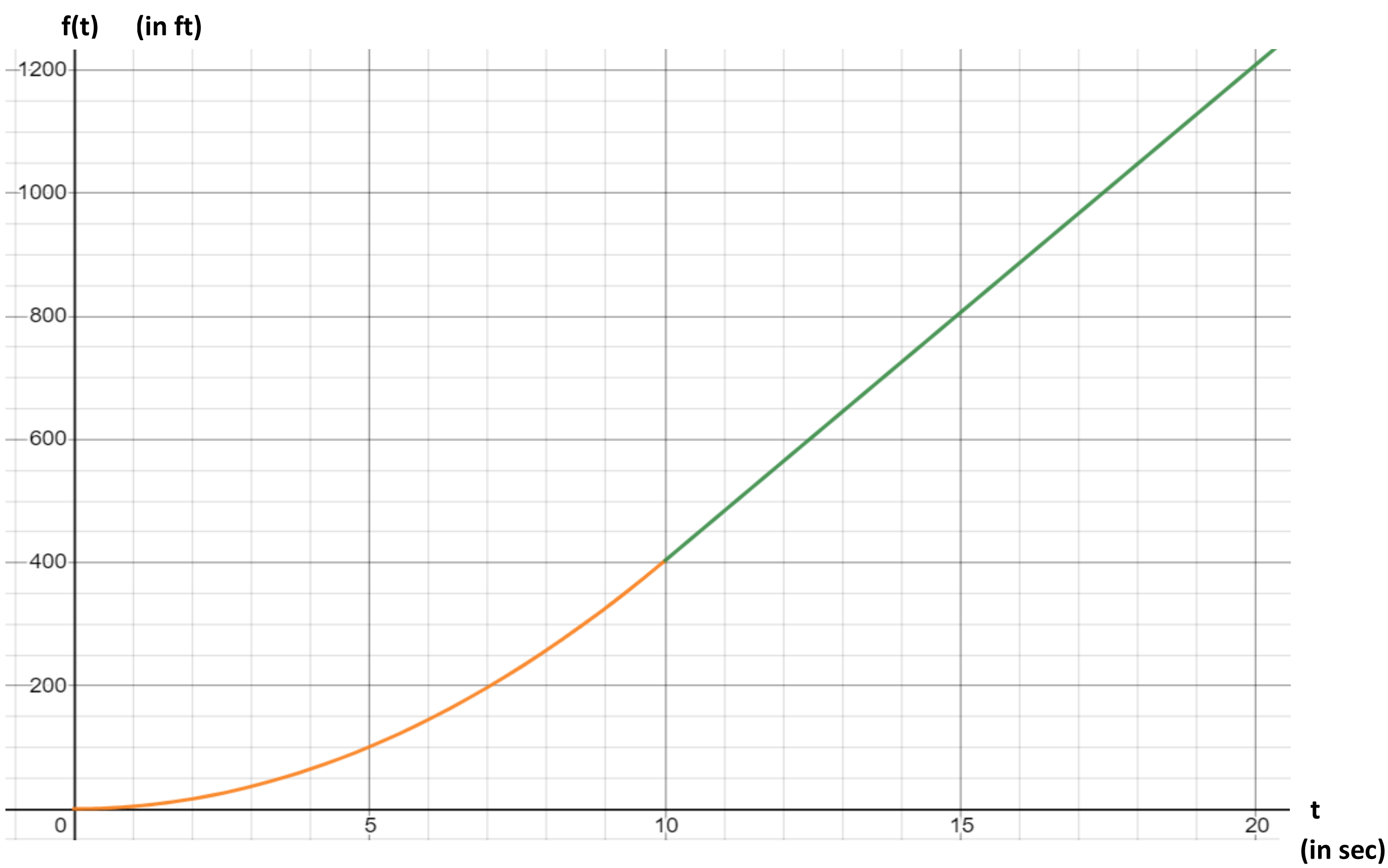
Introducing Continuity

A car travelling on a straight flat road goes from 0 to 55 miles per hour in 10 seconds and then travels at a constant speed of 55 miles per hour after that. In the first 10 seconds the acceleration (rate of change of velocity) is constant. If the foot markers are placed along the road so that the position at t = 0 seconds is 0 feet and so that the car is travelling in the direction of increasing foot markers (that is, the position is increasing), then the position function for the car is

$$s(t) = \begin{cases} \left(\frac{121}{30}\right) t^2 & for\ 0 \le t \le 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & for\ t > 10 \end{cases}$$ where t is in sec. and s(t) is in ft.

The graph of the position function follows.



Note from the graph that the two pieces of the piecewise defined position function

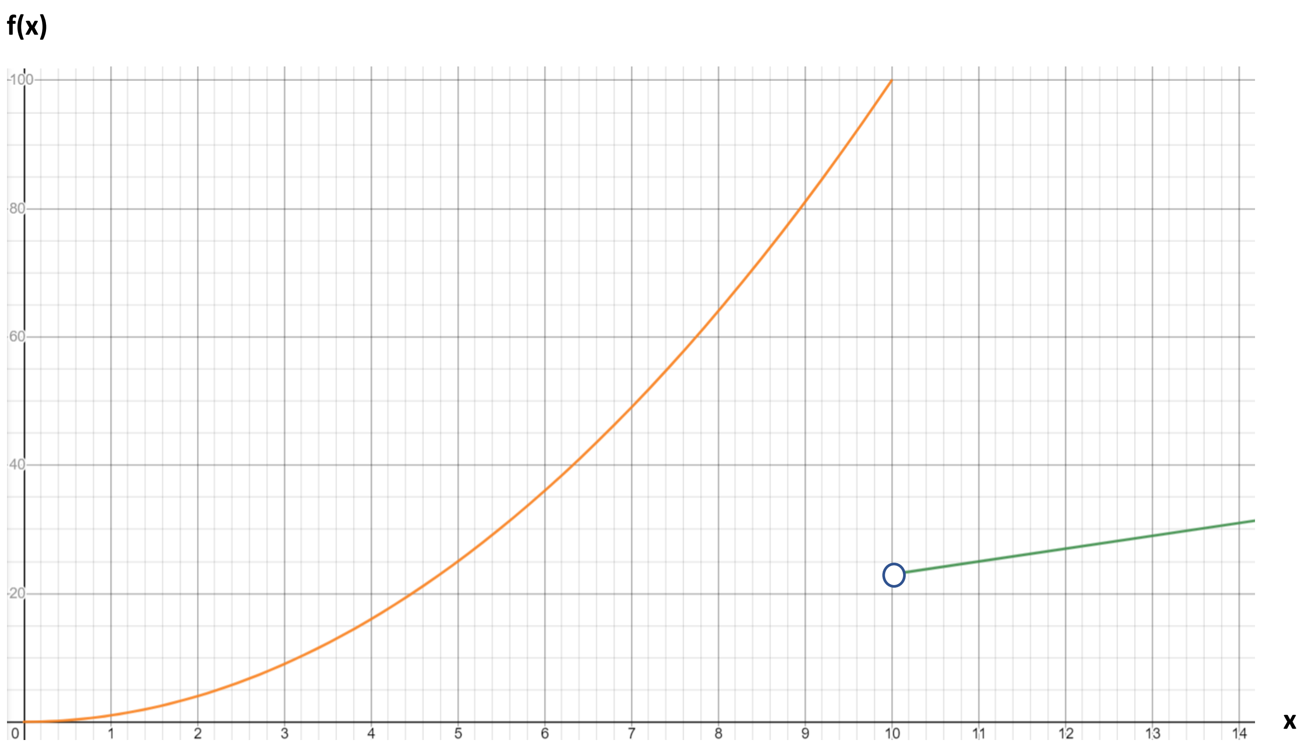
$$s(t) = \begin{cases} \left(\frac{121}{30}\right)t^2 & \text{for } 0 \le t \le 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & \text{for } t > 10 \end{cases}$$

connect. Is it surprising that they connect or is this something you would expect? Discuss with a neighbor.

Let's examine some other piecewise defined functions in the following examples.

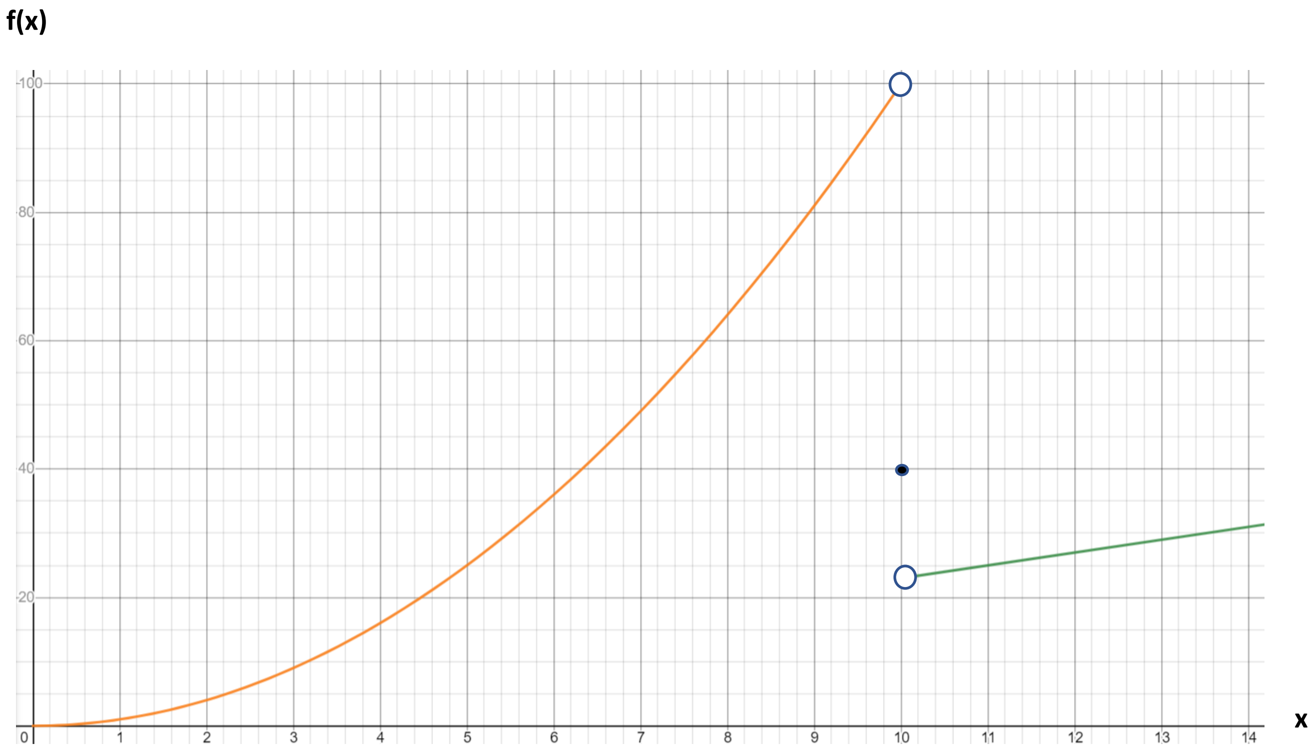
Example 1:

$$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 10 \\ 2x + 3 & \text{for } x > 10 \end{cases}$$



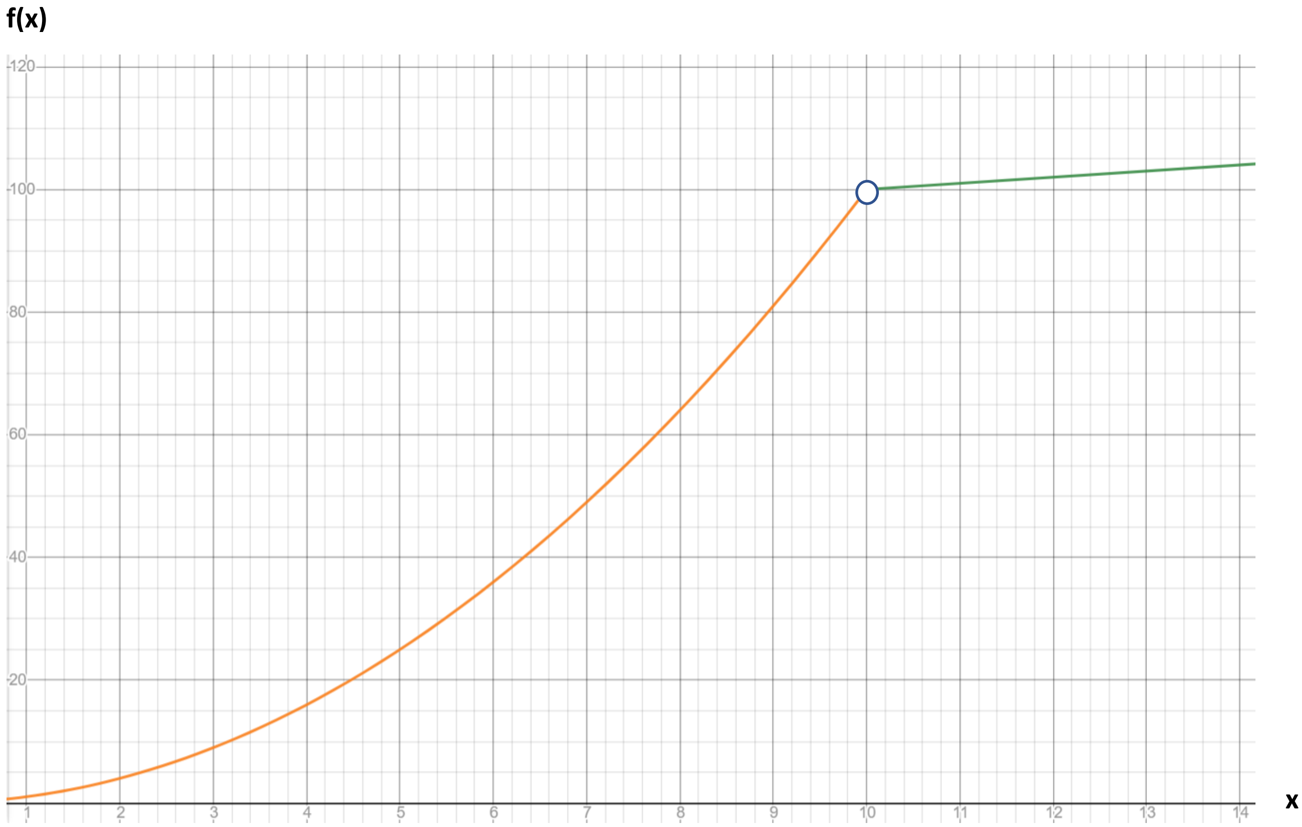
Example 2:

$$f(x) = \begin{cases} x^2 & for\ 0 \le x < 10 \\ 40 & for\ x = 10 \\ 2x + 3 & for\ x > 10 \end{cases}$$



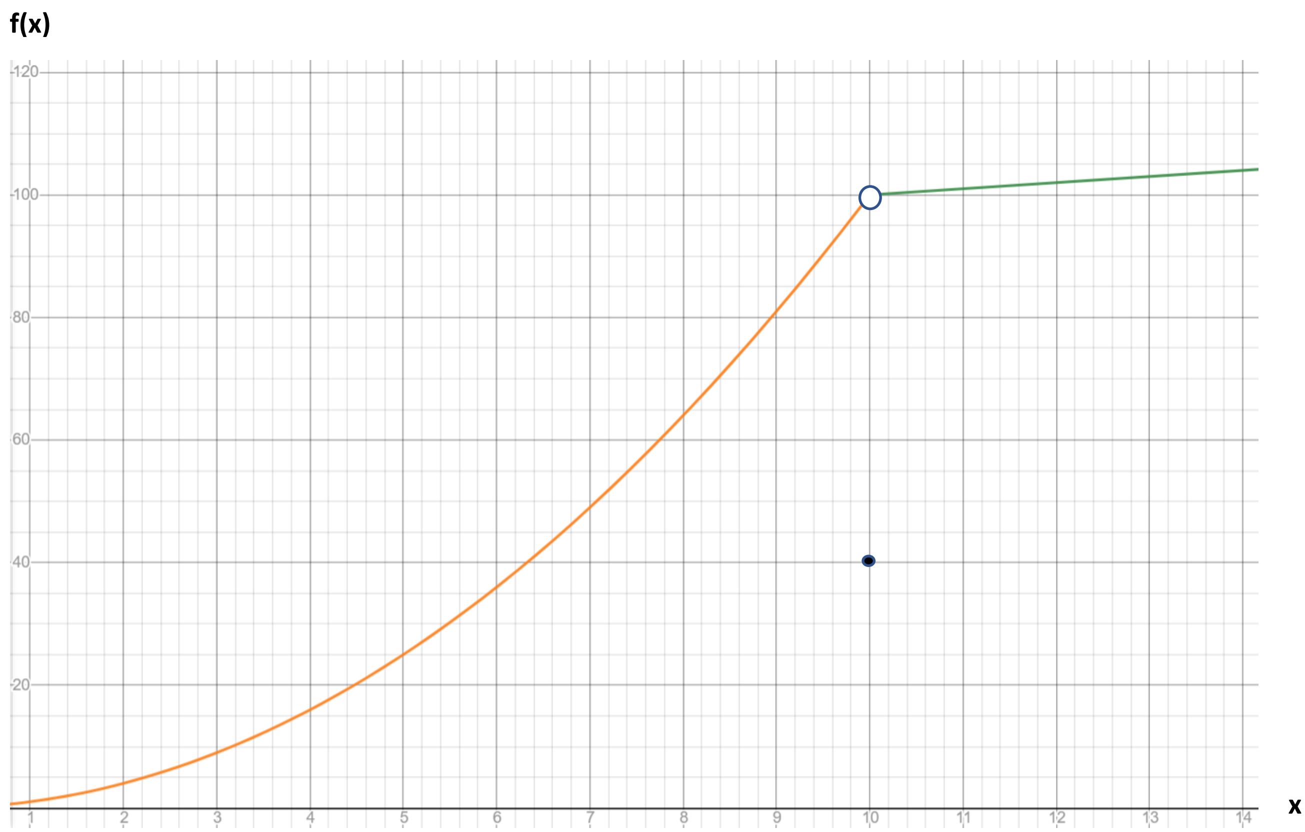
Example 3:

$$f(x) = \begin{cases} x^2 & for\ 0 \le x < 10 \\ 90 + x & x > 10 \end{cases}$$



Example 4:

$$f(x) = \begin{cases} x^2 & for\ 0 \le x < 10 \\ 40 & for\ x = 10 \\ 90 + x & for\ x > 10 \end{cases}$$



The position function s for the car scenario is continuous at $t = 10$ seconds, while the functions f in Examples 1, 2, 3, and 4 are not continuous at $x = 10$. Let's investigate the mathematical requirements, and how to express these requirements, for a function to be continuous at a point.

Problem 0) For the position function for our car scenario (see function and graph below), evaluate the items in parts a), b), c) and d).

a) s(10)

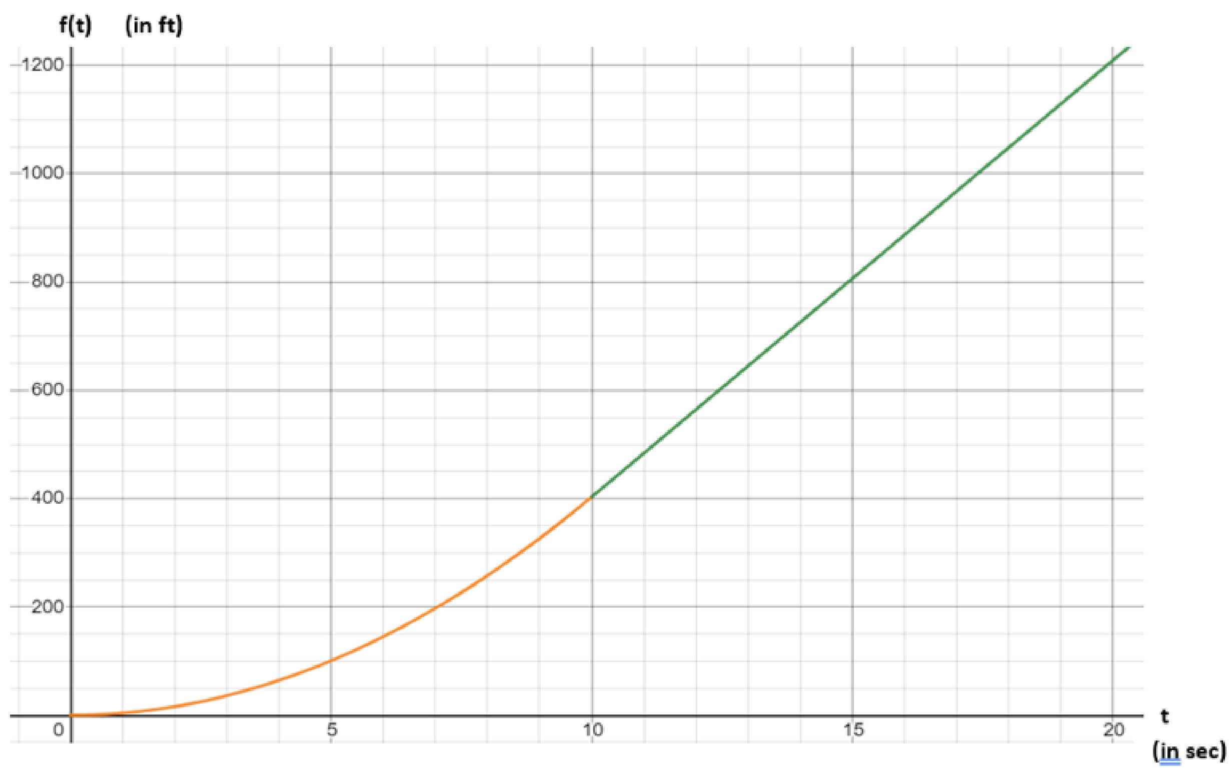
b) $\lim_{t \to 10^-} s(t)$

c) $\lim_{t \to 10^+} s(t)$

d) $\lim_{t \to 10} s(t)$

$$s(t) = \begin{cases} \left(\frac{121}{30}\right) t^2 & for\ 0 \le t \le 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & for\ t > 10 \end{cases}$$

where t is in sec. and s(t) is in ft.



Problem 1) For the function in Example 1 (see function and graph below), evaluate the items in parts a), b), c) and d).

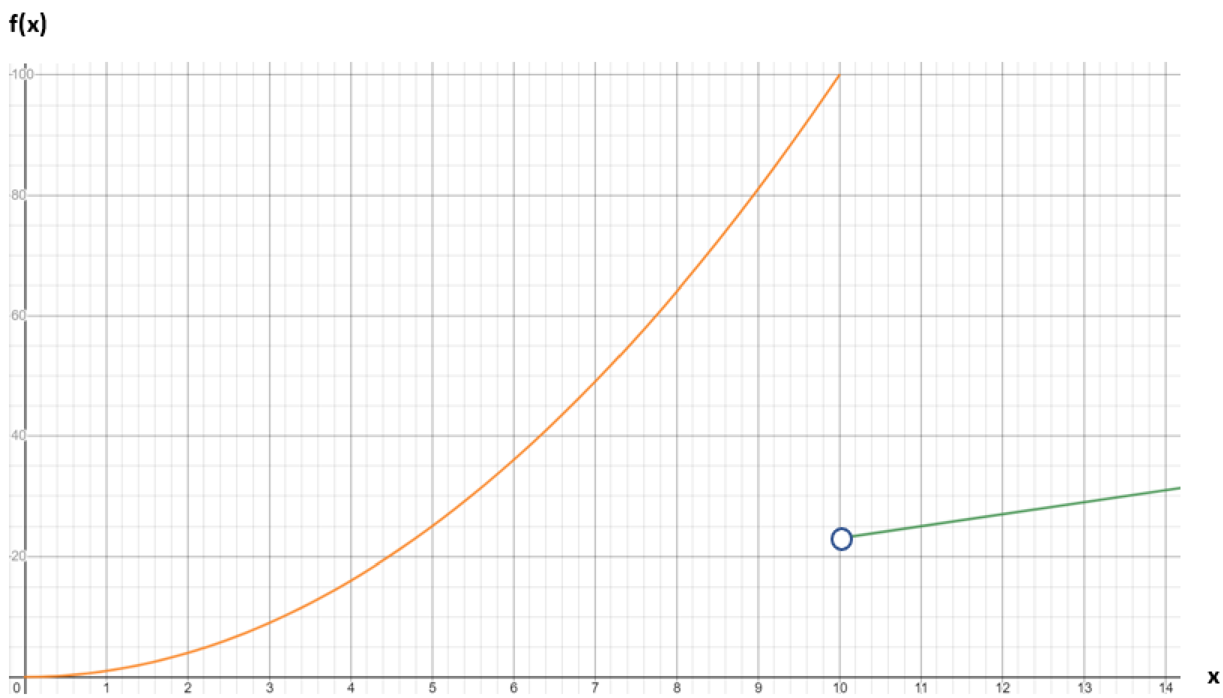
a) f(10)

b) $\lim_{x \to 10^-} f(x)$

c) $\lim_{x \to 10^+} f(x)$

d) $\lim_{x \to 10} f(x)$

$$f(x) = \begin{cases} x^2 & for\ 0 \le x \le 10 \\ 2x+3 & for\ x > 10 \end{cases}$$



<u>Problem 2)</u> For the function in Example 2 (see function and graph below), evaluate the items in parts a), b), c) and d).

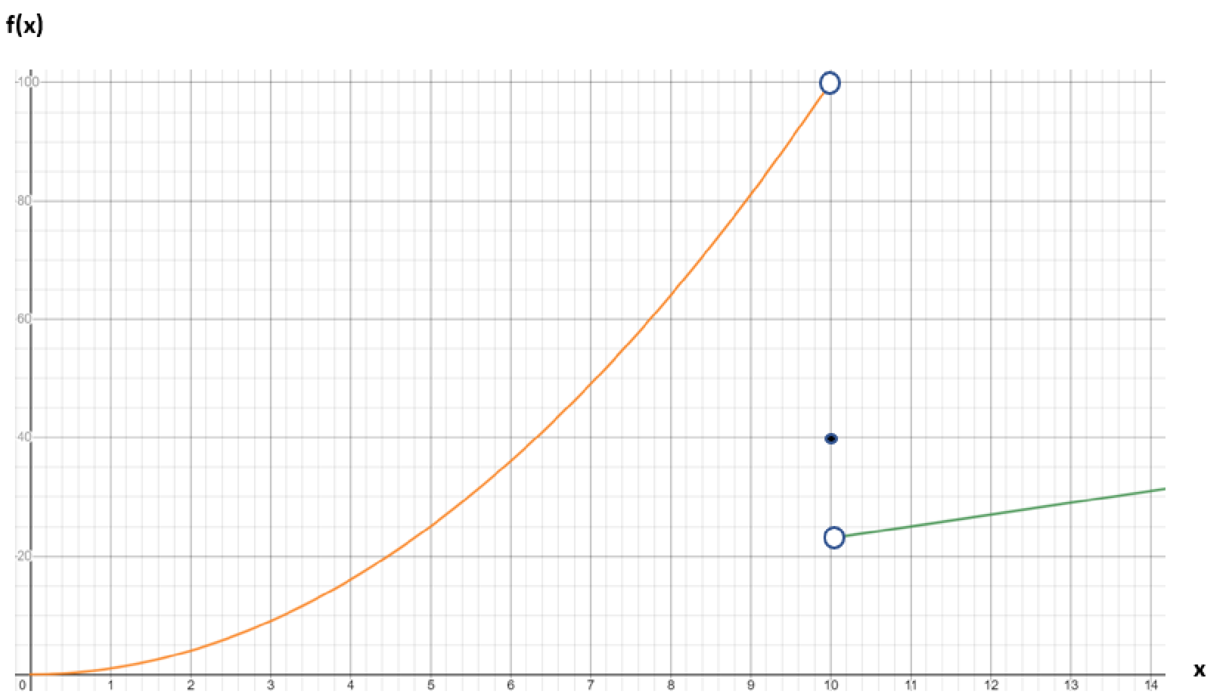
a) f(10)

b) $\lim_{x \to 10^-} f(x)$

c) $\lim_{x \to 10^+} f(x)$

d) $\lim_{x \to 10} f(x)$

$$f(x) = \begin{cases} x^2 & for\ 0 \le x < 10 \\ 40 & for\ x = 10 \\ 2x + 3 & for\ x > 10 \end{cases}$$



<u>Problem 3)</u> For the function in Example 3 (see function and graph below), evaluate the items in parts a), b), c) and d).

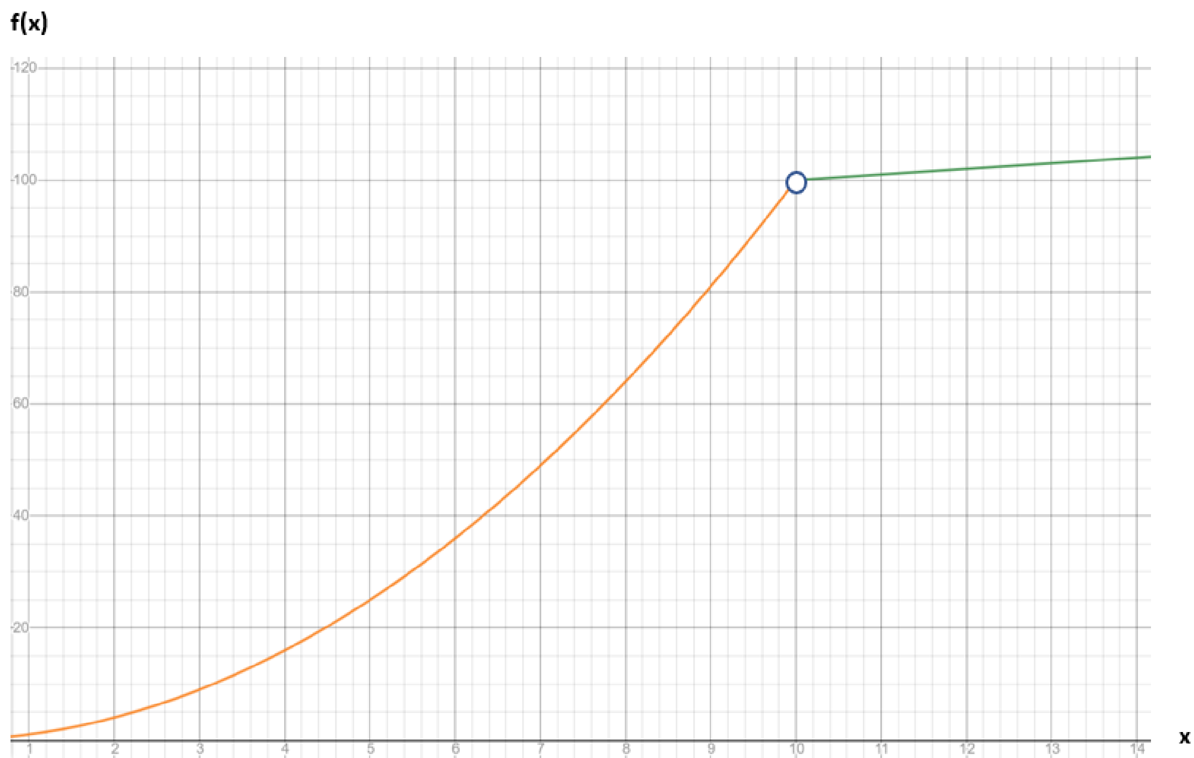
a) f(10)

b) $\lim\limits_{x \to 10^-} f(x)$

c) $\lim\limits_{x \to 10^+} f(x)$

d) $\lim\limits_{x \to 10} f(x)$

$$f(x) = \begin{cases} x^2 & for\ 0 \le x < 10 \\ 90 + x & x > 10 \end{cases}$$



Problem 4) For the function in Example 4 (see function and graph below), evaluate the items in parts a), b), c) and d).

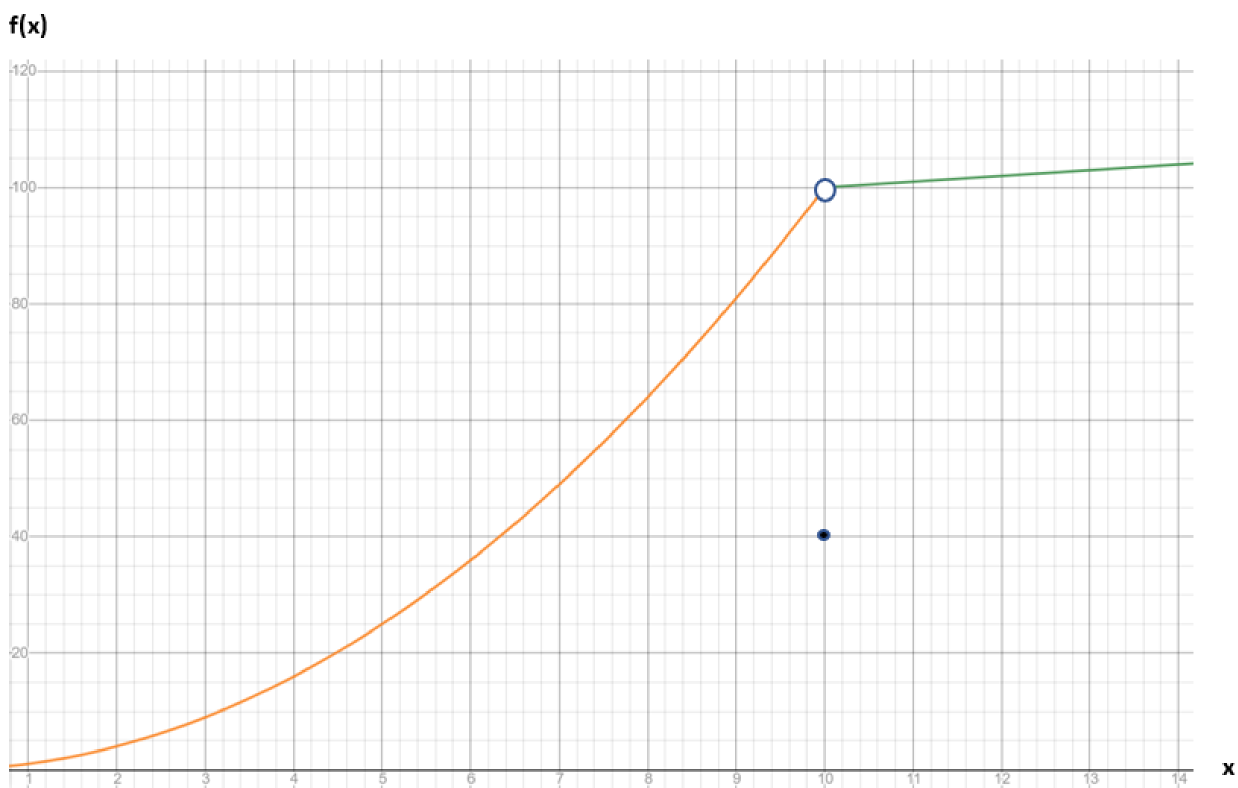
a) f(10)

b) $\lim_{x \to 10^-} f(x)$

c) $\lim_{x \to 10^+} f(x)$

d) $\lim_{x \to 10} f(x)$

$$f(x) = \begin{cases} x^2 & for\ 0 \le x < 10 \\ 40 & for\ x = 10 \\ 90 + x & for\ x > 10 \end{cases}$$



The function s in the car scenario (see Problem 0) is continuous at $t = 10$ while the functions f in Problems 1,2,3,4 are not continuous at $x = 10$. We say that the functions in Problems 1,2,3,4 are discontinuous at $x = 10$.

For a function f to be continuous at $x = 10$, three conditions must be met. Based on the Problems 0 – 4, and the results of questions a) – d), what do you think these three conditions are?

i)

ii)

iii)

Explain what these conditions mean in terms of the car scenario.

Why must the position function for the car be continuous at $t = 10$?

Which of the conditions fails for each of the functions in problems 1-4?

Generalize your previous result. For a function to be continuous at $x = a$, what three conditions must be met?

Teaching Notes

1) It is assumed that prior to this lesson, limits have been defined informally in a way similar to the following.

   Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a (page 83 from *Calculus, Early Transcendentals*, 8th edition by Stewart).

2) It is also assumed that prior to this lesson, students have some experience evaluating limits of the forms $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a} f(x)$ from inspecting the graph of f, and realize that for $\lim_{x\to a} f(x)$ to exist, both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$, must exist and be equal.

3) The intention in Problems 0) – 4) is that students use the graph of f to evaluate the limits, although if limit laws have been discussed they could use the symbolic representation of f.

4) Students often confuse $\lim_{x\to 2} f(x)$ with $f(2)$, for example. One contributing factor is not paying enough attention to the notation; if students consistently leave off lim as many do, then when they see $f(x)$ or when they see f(10), they may think $\lim_{x\to 10} f(x)$. It is important that they be encouraged to write the notation properly.

5) Students need to realize that a function can be continuous at one point and not at another; thus questions should be inserted into this activity which ask about continuity at values other than 10. Some students will not realize that in Example 2, for example, f(2) exists because it is not emphasized with a large dot as is the point (2, f(2)).

6) This activity shows examples of jump and removable discontinuities but not infinite discontinuities.