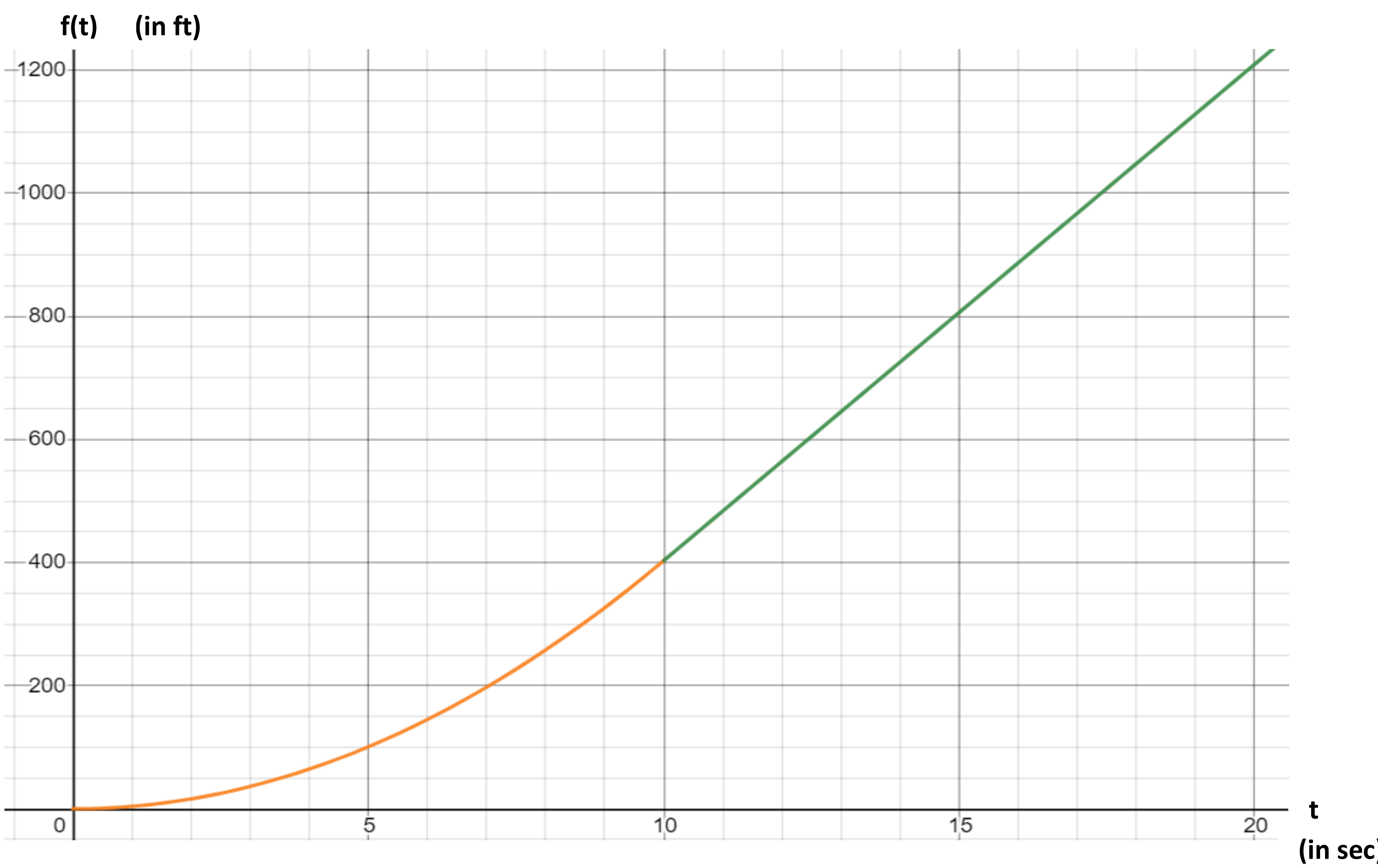
Introducing Differentiability

A car travelling on a straight flat road goes from 0 to 55 miles per hour in 10 seconds and then travels at a constant speed of 55 miles per hour after that. In the first 10 seconds the acceleration (rate of change of velocity) is constant. If the foot markers are placed along the road so that the position at t = 0 seconds is 0 feet and so that the car is travelling in the direction of increasing foot markers (that is, the position is increasing), then the position function for the car is

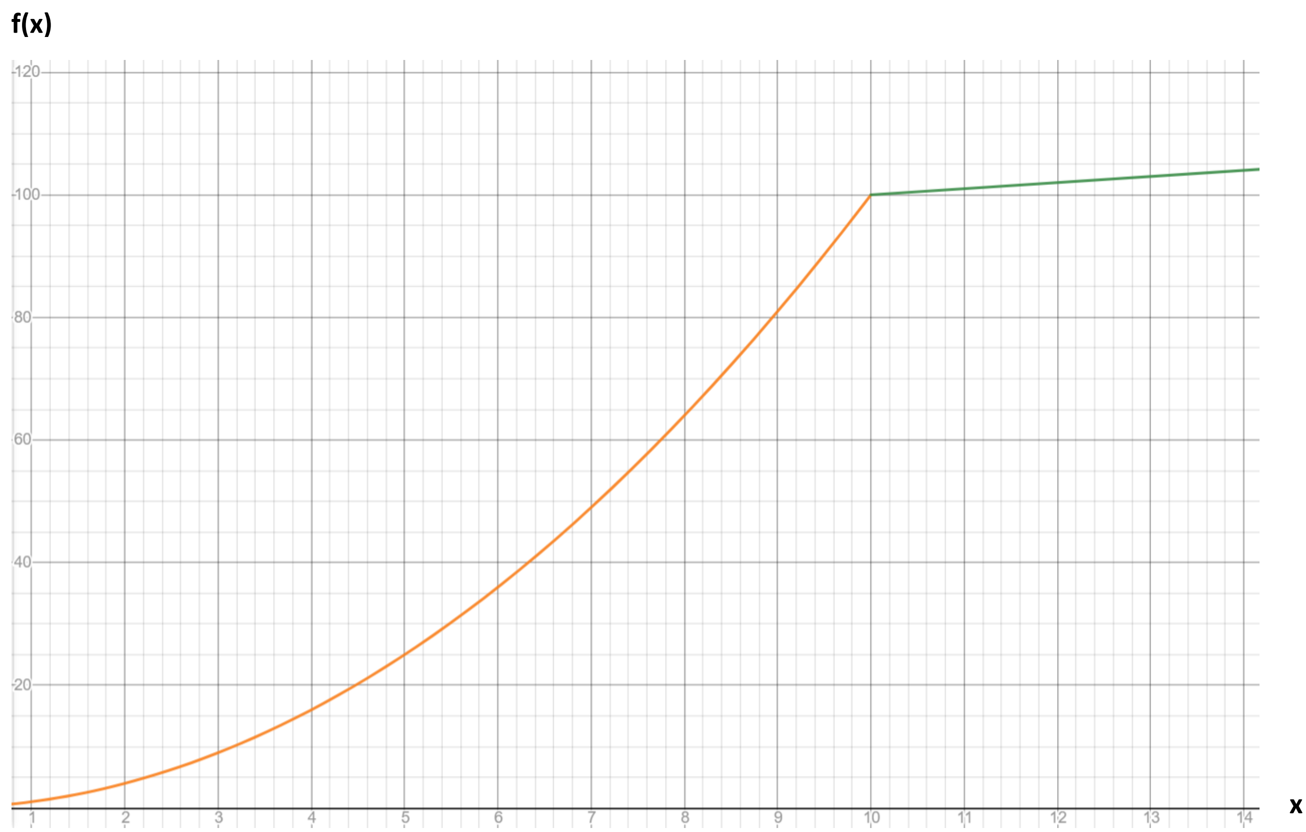
$$s(t) = \begin{cases} \left(\frac{121}{30}\right) t^2 & for\ 0 \le t \le 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & for\ t > 10 \end{cases}$$ where t is in sec. and s(t) is in ft.

The graph of the position function follows.

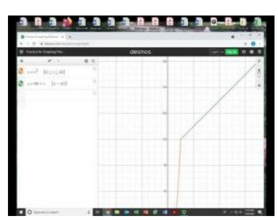


Notice from that graph the position function s is continuous at t = 10, but not only do the two pieces connect at $t = 10$, they also connect smoothly. In other words, the function s is smooth or differentiable at $t = 10$ in addition to being continuous at $t = 10$. This is in contrast with the following function which is not smooth (not differentiable) at $x = 10$, although it is continuous at $x = 10$.

$$f(x) = \begin{cases} x^2 & for\ 0 \le x \le 10 \\ 90 + x & x > 10 \end{cases}$$



In the short video embedded here,



we zoom in repeatedly on the above graph at the point $(10, f(10))$. As you watch the video, please observe that the sharp corner at the point $(10, f(10))$ remains, even though we are seeing the graph in a very small window after repeated zooming. This means that there is not a line tangent to the graph of f at the point $(10, f(10))$; the graph is not <u>locally linear</u>. Smooth, differentiable, locally linear are synonyms.

Exploring this symbolically, if there were a tangent line at $x = 10$, then its slope would be $\lim_{x \to 10} \frac{f(x)-f(10)}{x-10}$ which is $\lim_{x \to 10} \frac{f(x)-100}{x-10}$. We need to examine the left- and right- hand limits separately since f is a piece-wise defined function with the two pieces to the right and left of 10 having different equations.

The left-hand limit is: $\lim_{x \to 10^-} \frac{f(x)-100}{x-10} =$

$$\lim_{x \to 10^-} \frac{x^2-100}{x-10} = \lim_{x \to 10^-} \frac{(x-10)(x+10)}{x-10} = \lim_{x \to 10^-} (x+10) = 20$$

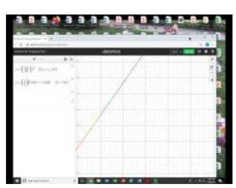
In other words, the slopes of the secant lines PQ approach 20 as Q approaches $P = (10, f(10))$ from the left.

The right-hand limit is: $\lim_{x \to 10^+} \frac{f(x)-100}{x-10} = \lim_{x \to 10^-} \frac{90+x-100}{x-10} = \lim_{x \to 10^-} \frac{x-10}{x-10} = \lim_{x \to 10^-} 1 = 1$

For points Q to the right of P, the secant lines PQ have a slope of 1.

So $\lim_{x \to 10} \frac{f(x)-f(10)}{x-10}$ does not exist, since the left-and right-hand limits are not equal. Thus, there is not a line tangent to the graph of f at the point $(10, f(10))$. The video supports this as it shows that after repeated zooming in at the point $(10, f(10))$, the graph of f does not appear to be locally linear; rather there continues to be a sharp corner at the point $(10, f(10))$.

Let's compare this with the graph of the position function of our car scenario. In the short video embedded here,



we zoom in repeatedly on the graph of the position function s at the point $(10, s(10))$. As you watch the video, please observe that the graph looks linear after repeated zooming. This line is the line tangent to the graph of s at the point $(10, s(10))$. The graph of the position function s at the point $(10, s(10))$ is locally linear, smooth, differentiable.

Exploring this symbolically, the slope of the line tangent to the graph of s at the point $(10, s(10))$ is $\lim_{t\to 10} \frac{s(t)-s(10)}{t-10}$ which is $\lim_{t\to 10} \frac{s(t)-403.\overline{3}}{t-10}$. We need to examine the left- and right- hand limits separately since s is a piece-wise defined function with the two pieces to the right and left of 10 having different equations.

The left-hand limit is: $\lim_{t\to 10^-} \frac{s(t)-403.\overline{3}}{t-10} =$

$$\lim_{t\to 10^-} \frac{4.0\overline{3}t^2-403.\overline{3}}{t-10} = \lim_{t\to 10^-} \frac{4.0\overline{3}(t^2-100)}{t-10} = \lim_{t\to 10^-} \frac{4.0\overline{3}(t-10)(t+10)}{t-10} = \lim_{t\to 10^-} 4.0\overline{3}(t+10) =$$

$$4.0\overline{3}(20) = 80.\overline{6}$$

In other words, the slopes of the secant lines PQ approach $80.\overline{6}$ as Q approaches $P = (10, s(10))$ from the left.

The right-hand limit is: $\lim_{t\to 10^+} \frac{s(t)-403.\overline{3}}{t-10} =$

$$\lim_{t\to 10^+} \frac{\left(\frac{1}{3}\right)(242t-1210)-403.\overline{3}}{t-10} = \lim_{t\to 10^+} \frac{80.\overline{6}t-806.\overline{6}}{t-10} = \lim_{t\to 10^+} \frac{80.\overline{6}(t-10)}{t-10} = \lim_{x\to 10^+} 80.\overline{6} = 80.\overline{6}$$

For points Q to the right of P, the secant lines PQ have a slope of $80.\overline{6}$.

So $\lim_{x\to 10} \frac{f(x)-f(10)}{x-10}$ exists and equals $80.\overline{6}$ since both the left-and right-hand limits equal $80.\overline{6}$. Thus, there is a line tangent to the graph of s at the point $(10, s(10))$, and its slope is $80.\overline{6}$. The video supports this as it shows that after repeated zooming in at the point $(10, s(10))$, the graph of s does appear to be locally linear.

Discuss: What does it mean in terms of the car ride for the graph of the position function to be smooth or differentiable at all points on the graph? What if someone speeded up abruptly and then slammed on the brakes frequently? Would this make the graph not be differentiable? Is it physically possible for a position function for a car travelling on a straight flat road to not be differentiable at a point?

Teaching Notes

1) A prerequisite for this lesson is that students have been introduced to the tangent problem and realize that the slope of the line tangent to the graph of f at the point $P = (a, f(a))$ is the limit of the slopes of secant lines PQ as Q approaches P. Using $(x, f(x))$ for Q, this can be written as:

   the slope of the line tangent to the graph of f at the point $P = (a, f(a))$ is $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

2) It should be discussed that the second function presented in this lesson is differentiable (smooth, locally linear) at all points except for $x = 10$. A good class demo would be to pick another point and zoom in on it.

3) A suggested assignment or small group activity would be for students to predict what will happen if we zoom in on each of the functions $f(x) = |x|$ and $f(x) = x^2$ at the point $(0, f(0))$ repeatedly, then zoom in to check their prediction, and finally confirm this result symbolically by taking the limits of difference quotients as illustrated in this lesson.