An Introduction to Antidifferentiation

A car travelling on a straight flat road goes from 0 to 55 miles per hour in 10 seconds and then travels at a constant speed of 55 miles per hour after that. In the first 10 seconds the acceleration (rate of change of velocity) is constant. Previously we stated that the position function for this motion is

$$s(t) = \begin{cases} \left(\frac{121}{30}\right)t^2 & for\ 0 \leq t \leq 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & for\ t > 10 \end{cases}$$

where t is in seconds and s(t) is in feet, and we used this position function to find the speedometer reading at various times, to introduce the concept of a limit. **But how do we arrive at this position function? Let's explore this question. In doing so, we will introduce the idea of antidifferentiation.**

Let's start with finding the position function on the time interval $t > 10$. We are given that at 10 seconds and afterwards, the velocity is 55 $\frac{mi}{hr}$. But $55\frac{mi}{hr} = \frac{242}{3}\frac{ft}{s}$.

So, when $t > 10$ seconds, the velocity is given by: $v(t) = \frac{242}{3}\frac{ft}{s}$ However, we need the position function $s(t)$ instead of the velocity function. We know the derivative of the position function is the velocity function, that is $s'(t) = v(t)$. So we need to find a function $s(t)$ whose derivative is $v(t) = \frac{242}{3}$.

Please answer: If the independent variable is t, find a function that you can take the derivative of to get the constant $\frac{242}{3}$?

Did you suggest $s(t) = \frac{242}{3}t$? $\boldsymbol{\frac{242}{3}t}$ **is an <u>antiderivative</u> of** $\boldsymbol{\frac{242}{3}}$ **because the derivative of** $\boldsymbol{\frac{242}{3}t}$ **with respect to t is** $\boldsymbol{\frac{242}{3}}$**.**

However, isn't the derivative of $s(t) = \frac{242}{3}t + 5$ also equal to $\frac{242}{3}$, or for that matter, couldn't there be any constant in place of the 5 and the derivative still be $\frac{242}{3}$?

So, we see that if a function has an antiderivative, it has infinitely many. All of the antiderivatives differ by a constant. **We say that the <u>most general</u> <u>antiderivative</u> of** $\boldsymbol{\frac{242}{3}}$ **is** $\boldsymbol{\frac{242}{3}t + c}$ **where c could be any real number.** So

$$s(t) = \frac{242}{3}t + c \quad for\ t > 10\ . \text{ (Equation 1)}$$

But to meet the specific conditions of our problem, there is only one specific value of c that applies. We will be able to determine this value monentarily.

Let's consider the first ten seconds. In the first ten seconds, the acceleration is constant, that is, the velocity is changing at a constant rate. So for $0 \leq t \leq 10$, $a(t) = k$. We know the derivative of the velocity function is the acceleration function, that is $v'(t) = a(t)$. So, to find the velocity function $v(t)$, we need to find a function *whose derivative is a(t) =* k.

Please answer: If the independent variable is t, find a function whose derivative is the constant k.

Did you suggest $v(t) = kt$?

kt is an <u>antiderivative</u> of k because the derivative of kt with respect to t is k. However, there are infinitely many antiderivatives of k, all differing by a constant. **The <u>most general antiderivative</u>** of k is $\mathrm{kt} + c$. Thus,

$$v(t) = \mathrm{kt} + c \text{ for } 0 \leq t \leq 10\ . \text{ (Equation 2)}$$

But to meet the specific conditions of our problem, there is only one value of c that applies. To find this value of c, we use the fact the velocity at 10 seconds is 55 miles per hour or $\frac{242}{3}\frac{ft}{s}$. That is, $v(10) = \frac{242}{3}$. Using equation 2, we have $v(10) = k \cdot 10 + c$. So

$$10k + c = \frac{242}{3}. \text{ (Equation 3)}$$

We will come back to this momentarily.

We also know that the velocity at *t = 0* seconds is $0\,\frac{ft}{s}$ *so using equation 2,* $v(0) = k\,(0) + c = 0$. Thus $0 + c = 0$, which makes $c = 0$. So equation 2 $v(t) = \mathrm{kt} + c$ for $0 \leq t \leq 10$ becomes

$$v(t) = \mathrm{kt} \text{ for } 0 \leq t \leq 10. \text{ (Equation 4)}$$

Going back to equation 3, we now know that $c = 0$, so equation 3 becomes $10k + 0 = \frac{242}{3}$. and thus $k = \frac{242}{30} = \frac{121}{15}$. Thus equation 4 becomes

$$v(t) = \frac{121}{15}\mathrm{t} \text{ for } 0 \leq t \leq 10. \text{ (Equation 5)}$$

This is the velocity function for the first 10 seconds. To find the position function $s(t)$ for the first 10 seconds, we need to find the most general antiderivative of $v(t) = \frac{121}{15}\mathrm{t}$ and then find the appropriate constant to meet the conditions of the problem.

Please answer: If the independent variable is t, find a function whose derivative is $\frac{121}{15}\mathrm{t}$.

Did you suggest $\frac{121}{15}\frac{t^2}{2}$?

$\frac{121}{15}\frac{t^2}{2}$ is an antiderivative of $\frac{121}{15}\mathrm{t}$ because the derivative of $\frac{121}{15}\frac{t^2}{2}$ is $\frac{121}{15}\mathrm{t}$. So $s(t) = \frac{121}{15}\frac{t^2}{2} + \mathrm{c}$ for $0 \le t \le 10$. This simplifies to

$$s(t) = \frac{121}{30}t^2 + \mathrm{c} \quad 0 \le t \le 10. \text{ (Equation 6)}$$

While this does not have to be the case, it simplifies things for us to start measuring position from where the car was located at $t = 0$ seconds, in other words to make 0 feet be the position at $t = 0$ seconds. Thus $s(0) = 0$. Using equation 6, we have $s(0) = \frac{121}{30}0^2 + \mathrm{c} = 0$ which makes $c = 0$. So

$$s(t) = \frac{121}{30}t^2 + 0 \text{ for } 0 \le t \le 10. \text{ (Equation 7)}$$

is the position function for the first 10 seconds.

Earlier (equation 1) we found the position function for $t > 10$ seconds to be $s(t) = \frac{242}{3}t + c$. To find the value of c that meets the conditions of this problem, we use the fact that for the position function to be continuous at $t = 10$ seconds, $\lim_{t\to 10^+} s(t)$ must equal $\lim_{t\to 10^-} s(t)$. Using the expressions for $s(t)$ previously found for the time intervals before and after 10 seconds (equations 1 and 7) we have:

$$\lim_{t\to 10^+} s(t) = \lim_{t\to 10^+}\left(\frac{242}{3}t + c\right) = \frac{242}{3}\cdot 10 + c$$

$$\lim_{t\to 10^-} s(t) = \lim_{t\to 10^-}\left(\frac{121}{30}t^2\right) = \frac{121}{30}\cdot 10^2 = \frac{1210}{3}$$

For these left- and right-hand limits to be equal, the following equation must be satisfied: $\frac{242}{3} \cdot 10 + c = \frac{1210}{3}$. Solving for c gives $c = \frac{-1210}{3}$. Thus equation 1 becomes $s(t) = \frac{242}{3}t - \frac{1210}{3} \quad for\ t > 10$ which can be written as

$$s(t) = \left(\frac{1}{3}\right)(242t - 1210) \ \ for\ t > 10$$

Thus, the position function for the scenario of a car travelling on a straight flat road, going from 0 to 55 miles per hour in 10 seconds with constant acceleration, and then traveling at a constant speed of 55 miles per hour after 10 seconds is

$$s(t) = \begin{cases} \left(\frac{121}{30}\right)t^2 & for\ 0 \le t \le 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & for\ t > 10 \end{cases}$$ with t in seconds and $s(t)$ in feet.

provided the foot markers are placed along the road so that the position at t = 0 seconds is 0 feet and so that the car is travelling in the direction of increasing foot markers (that is, the position is increasing.)

Teaching Notes

1) Page 1 suffices to motivate the need for antidifferentiation and introduce the topic.
2) The idea of most general antiderivative which was introduced in regard to equation 1, in contrast to antiderivative, can be further illuminated by a discussion of the fact that where the motion along a line begins doesn't affect the velocity. The motion could be exactly the same (going from 0 to 55 mph in 10 seconds and then travelling at a constant speed of 55 mph) starting from various points on the line. The constant in the position function depends upon where along the line the motion is occurring, not the velocity.)