Motivating the Fundamental Theorem of Calculus

Recall that a definite integral $\int_a^b f(x)dx$ is defined to be the limit of a sum $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i^*})\Delta x$, and this gives the area between the graph of $y = f(x)$ and the x-axis on the interval $[a,b]$ provided $y = f(x) > 0$ on $[a,b]$.

A car travelling on a straight flat road goes from 0 to 55 miles per hour in 10 seconds and then travels at a constant speed of 55 miles per hour after that. In the first 10 seconds the acceleration (rate of change of velocity) is constant. If the foot markers are placed along the road so that the position at t = 0 seconds is 0 feet and so that the car is travelling in the direction of increasing foot markers (that is, the position is increasing), then we found the position function for the car to be

$$s(t) = \begin{cases} \left(\frac{121}{30}\right)t^2 & for\ 0 \le t \le 10 \\ \left(\frac{1}{3}\right)(242t - 1210) & for\ t > 10 \end{cases}$$ where t is in sec. and s(t) is in ft.

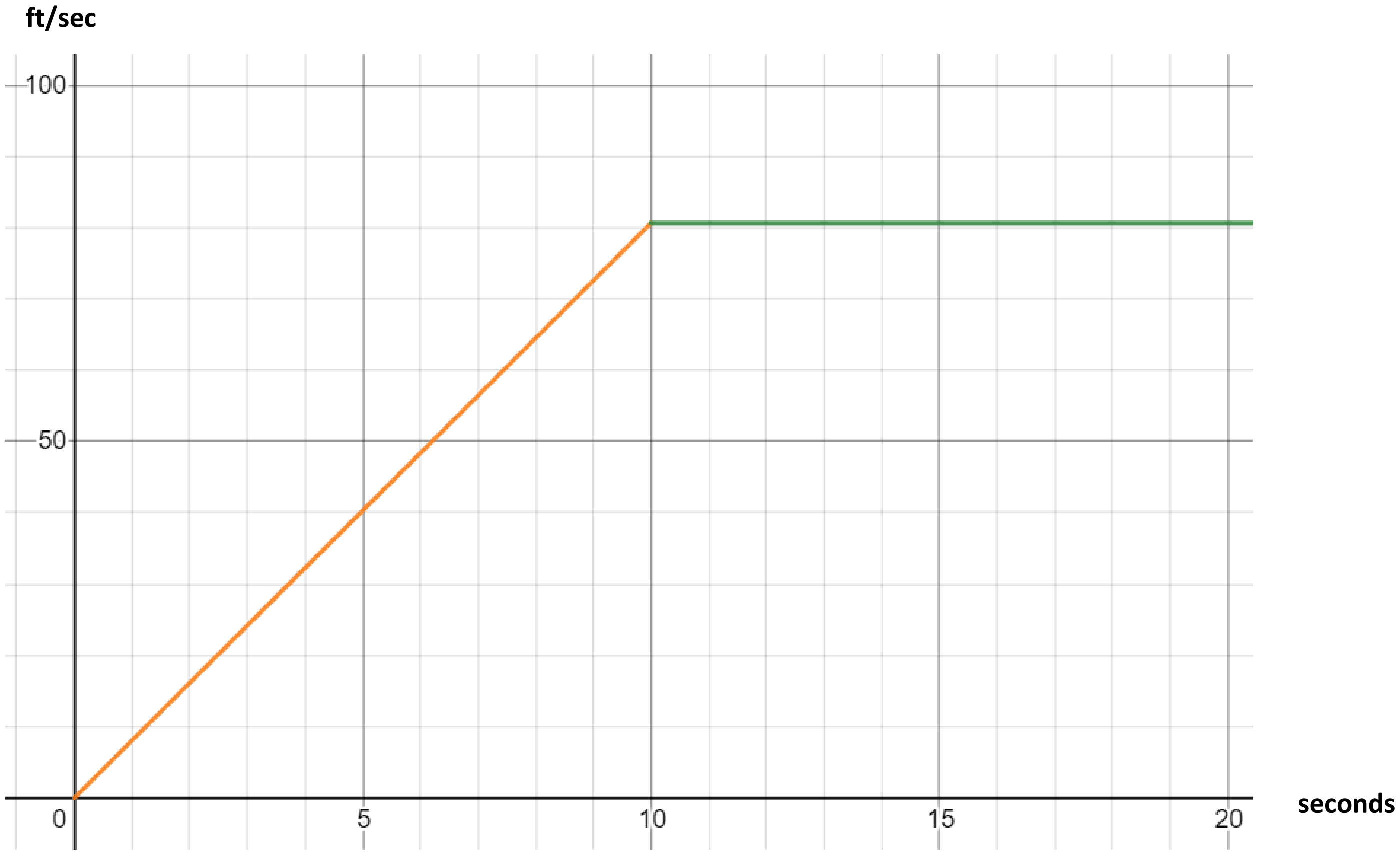
We also found the velocity function to be

$$v(t) = \begin{cases} \left(\frac{121}{15}\right)t & for\ 0 \le t \le 10 \\ \frac{242}{3} & for\ t > 10 \end{cases}$$ where t is in sec. and v(t) is in ft/sec.

The speed function is the same as the velocity function since the position is increasing and therefore the rate of change of position (velocity) is positive. Since speed is the absolute value of velocity and the velocity is positive, speed and velocity are the same in this example. Thus, the speed of the car at time t is given by the following.

$$speed(t) = \begin{cases} \left(\frac{121}{15}\right)t & for\ 0 \le t \le 10 \\ \frac{242}{3} & for\ t > 10 \end{cases}$$ where t is in sec. and speed(t) is in ft/sec.

The graph of the velocity function follows. This is also the graph of the speed function.



Note that the velocity is linear for the first ten seconds (in other words, the rate of change of velocity (acceleration) is constant), and then after 10 seconds the velocity is constant.

Compute the area between the velocity function graph and the horizontal axis on the interval [4,10]. Note that you are just computing the area of a trazepoid and you have used a formula from your prior study of geometry for the area of a trazepoid, $A = \frac{(b_1+b_2)h}{2}$ Make sure you use the equation for $v(t)$ to get exact values of the bases, b_1 and b_2, as you will be able to read only approximate values from the graph. However, you should use these approximate values from the graph to confirm the reasonableness of your calculated values for the bases.

Did you get $\frac{1694}{5}$?

The area of a trapezoid is $\frac{(b_1+b_2)\cdot h}{2}$ The height is 10 – 4 or 6, and the bases and v(4) and v(10). Now $v(4) = \left(\frac{121}{15}\right)\cdot 4 = \frac{484}{15}$ and $v(10) = \left(\frac{121}{15}\right)\cdot 10 = \frac{1210}{15}$.

So, the area of the trapezoid is $\frac{\left(\frac{484}{15}+\frac{1210}{15}\right)\cdot 6}{2} = \frac{1694}{5}$

But let's pay attention to the units. The height represents 10 seconds – 4 seconds or 6 seconds. The bases represent $\frac{484}{15}\frac{ft}{s}$ and $\frac{1210}{15}\frac{ft}{s}$. So the area represents

$$\frac{\left(\frac{484}{15}\frac{ft}{s} + \frac{1210}{15}\frac{ft}{s}\right)\cdot 6s}{2} = \frac{1694}{5}ft$$

So the area between our velocity graph and the horizontal axis on the interval $[\mathbf{4}, \mathbf{10}]$ really represents $\frac{1694}{5}ft$. We will call this Finding 1) and will come back to it shortly.

Now using the position function for the motion of the car, compute s(10) – s(4), paying attention to units, and think about what this means in practical terms.

Did you get $\frac{1694}{5}ft$ for $s(10) - s(4)$?

$$s(10) = \left(\frac{121}{30}\right) \cdot 10^2 = \frac{12100}{30} \text{ and } s(4) = \left(\frac{121}{30}\right) \cdot 4^2 = \frac{1936}{30} \text{ so}$$

$$s(10) - s(4) = \frac{12100}{30} - \frac{1936}{30} = \frac{10164}{30} = \frac{1694}{5}$$

But let's pay attention to the units. The units of $s(t)$ are feet, so the $\frac{1694}{5}$ represents $\frac{1694}{5}ft$

So the difference in the ending and beginning positions of the car in the time interval $[4, 10]$ (which is the total distance travelled in this case since the motion is such that position is increasing) is $\frac{1694}{5} feet$. We will call this Finding 2).

> What do you notice about Finding 1 and Finding 2? Do you think this is always going to be the case and why?

For the particular example we are examining, the total area between the graph of the velocity function and the horizontal axis on a chosen interval of time is the total distance travelled during that interval of time. In this example, a 1 by 1 square unit of area ■ represents 1 ft because the width is 1 second and the length is 1 ft/second, so the product is $(1\ second) \cdot \left(1\ \frac{foot}{second}\right) = 1\ foot$. If the velocity function changes, this will still be true (as long as time is in seconds and distance in feet; if the units are different, the principle is the same.) Even if the velocity function is negative, the total area between the graph of the velocity function and the horizontal axis on the interval $[a, b]$ is the total distance travelled.

> Considering the case where $v(t) > 0$, express the relationship above using integral notation.

Did you write: In the case of $v(t) > 0$, we found that $\int_a^b v(t)dt = s(b) - s(a) \; where \; s'(t) = v(t)$?

Explanation: We found that the area between the graph of $v(t)$ and the horizontal axis on the interval $[a, b]$ is the total distance travelled. But when $v(t) > 0$, then $\int_a^b v(t)dt$ is exactly the area between the graph of $v(t)$ and the horizontal axis on the interval $[a, b]$. When $v(t) > 0$, it is also true that $s(b) - s(a)$ is the total distance travelled. Also note that since $v(t) = s'(t)$, then $s(t)$ is an antiderivative of $v(t)$. Thus, in the case of $v(t) > 0$, the statement "the total area between the graph of the velocity function and the horizontal axis on a chosen interval is the total distance travelled during that interval" can be written as

$\int_a^b v(t)dt = s(b) - s(a) \; where \; s'(t) = v(t)$.

This can be generalized to any function $f(x)$ for which $f(x) > 0$ on $[a, b]$. In other words,

$$\int_a^b \boldsymbol{f(x)dx = F(b) - F(a) \; where \; F'(x) = f(x)}.$$

This is a statement of the Fundamental Theorem of Calculus (Part 2). It turns out that the Fundamental Theorem of Calculus (Part 2) is also true even if $f(x)$ is not always greater than 0 on the interval $[a, b]$. Where $f(x)$ is a velocity function $v(t)$, regardless of whether $v(t) > 0$ or not, $\int_a^b v(t)dt$ is the net area between the graph of $y = f(t)$ and the horizontal axis on $[a, b]$, meaning the total area above the horizontal axis minus the total area below the horizontal axis, and $s(b) - s(a)$ is the net distance travelled (the total distance travelled in the positive direction minus the total distance travelled in the negative direction.)

Teaching Notes

1) Prior to this lesson students should have computed Riemann sums to approximate areas under a curve, found the exact area under a curve by computing $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i^*})\Delta x$, and have been exposed to the fact that the notation $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i^*})\Delta x$ is shortened to be $\int_a^b f(x)dx$, and thus understanding the definite integral as a limit of a sum and understanding that when a function is positive, the definite integral gives the exact area between the graph of the function and the x-axis on the interval $[a, b]$. They should also realize that $\int_a^b f(x)dx$ gives net area between the graph of the function and the x-axis on the interval $[a, b]$ when it is not true that $f(x) > 0$ throughout $[a, b]$. It should be emphasized that a definite integral MEANS something very different from an indefinite integral $\int f(x)dx$ which is defined to be the most general antiderivative of f. Definite integrals have numerical values while indefinite integrals are functions, thus emphasizing the fact that they mean different things. But why are they both called integrals and have such similar symbolism if definite and indefinite integrals mean different things? Obviously there has to be some connection; otherwise using such similar notation for such dissimilar things would be foolhardy. The connection is the fundamental theorem of calculus. This is the backdrop which sets the stage for the lesson, Motivating the Fundamental Theorem of Calculus. The fundamental theorem of calculus gives an important connection between differential and integral calculus, between antiderivatives (indefinite integrals) and definite integrals.

2) This discussion in this lesson is about motivating part 2 of the Fundamental Theorem of Calculus, and does not refer to part 1.