Solutions to Average and Instantaneous Speeds: Introducing Limits

EXERCISE 1: Place the car in its proper position along the road at the following times: t = 0, t = 2, t = 4, t = 6, t = 8, t = 10, t = 12, t = 14, t = 16, t = 18. That is, evaluate s(0), s(2), s(4), s(6), s(8), s(10), s(12), s(14), s(16), s(18) and mark the location at the requested times on a number line.

$s(0) = 0\,ft.$

$s(2) = \frac{242}{15} ft. = 16.1\overline{3}\,ft.$

$s(4) = \frac{968}{15}\,ft. = 64..5\overline{3}\,ft.$

$s(6) = \frac{726}{5}\,ft. = 145.2\,ft.$

$s(8) = \frac{3872}{15}\,ft. = 258.1\overline{3}\,ft.$

$s(10) = \frac{1210}{3}\,ft. = 403.\overline{3}\,ft.$

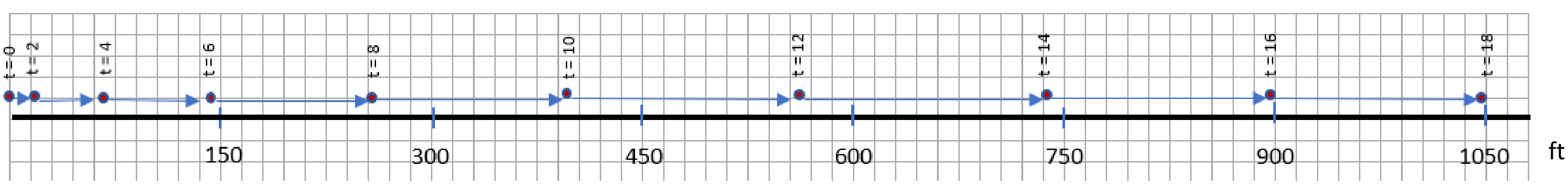
$s(12) = \frac{1694}{3}\,ft. = 564.\overline{6}\,ft,$

$s(14) = 726\,ft.$

$s(16) = \frac{2662}{3} ft. = 887.\overline{3}\,ft.$

$s(18) = \frac{3146}{3} ft. = 1048.\overline{6}\,ft.$

The numberline represents the road. Each gridline represents 15 ft. The dots represent the position of the car on the road at two second intervals as labeled. The arrows represent the direction of motion. Notice that the car travels increasing distances each subsequent 2 second time interval prior to 10 seconds and after 10 seconds the car travels the same distance in each 2 second time interval. Why is this true?



EXERCISE 2: Calculate the average speed of the car during each of the time intervals in the table below. Set your calculator to floating point mode and write all the decimal digits that appear in your calculator display or use www.wolframalpha.com and truncate after 8 digits to the right of the decimal point. (Do not write down intermediate answers and re-enter these in your calculator; instead let the calculator compute the entire expression for average speed to avoid roundoff error propagation and calculator entry error.)

Time Interval	Average speed in the time interval (in ft per second)	Time Interval	Average speed in the time interval (in ft per second)
[1, 1.0001]	8.06707 ft/sec	[.9999,1]	8.066263333 ft/sec
[1, 1.00001]	8.066707 ft/sec	[.99999,1]	8.066626333 ft/sec
[1, 1.000001]	8.0666707 ft/sec	[.999999,1]	8.066662633 ft/sec
[1, 1.0000001]	8.06666707 ft/sec	[.9999999,1]	8.066666263 ft/sec

Sample calculation: $\frac{s(1.0001)-s(1)}{1.0001-1} = \frac{\left(\frac{121}{30}\right)\cdot 1.0001^2-\left(\frac{121}{30}\right)\cdot 1^2}{1.0001-1} = \frac{\left(\frac{121}{30}\right)(1.0001^2-1^2)}{1.0001-1} = 8.06707\ \frac{ft}{sec}$

Note that it is helpful to use the 2nd entry button on your calculator to bring up prior calculations and edit them by inserting extra zeros rather than re-entering expressions of this form $\frac{\left(\frac{121}{30}\right)(1.0001^2-1^2)}{1.0001-1}$.

EXERCISE 3: Based on the average speed calculations in Exercise 2, predict the instantaneous speed of the car at t = 1 second. Give your answer in ft per second and then convert from ft/sec to mi/hr. (There are 5280 ft in a mile.) You have predicted the speedometer reading of the car at t = 1 second. Is this reasonable?

Initial predictions will vary in terms of precision. Using the series of questions appearing in the Teaching Notes, I seek to help students refine their precision and language. My goal is for students to understand the following from doing and discussing exercises 2 and 3 and to use language such as the following: As the time intervals in column 1 shrink down upon the instant in time $t = 1$ from the right, the average speeds in column 2 seem to be approaching $8.0\overline{6}\frac{ft}{s}$. As the time intervals in column 3 shrink down upon the instant in time $t = 1$ from the left, the average speeds in column 4 seem to be approaching $8.0\overline{6}\frac{ft}{s}$. Thus, the limit of the average speeds in time intervals shrinking down upon the instant in time $t = 1$ seems to be $8.0\overline{6}\frac{ft}{s}$. Therefore, I predict that the instantaneous speed at $t = 1$ (speedometer reading at $t = 1$) is $8.0\overline{6}\frac{ft}{s}$.

Converting our answer from ft/s to mi/hr: $8.0\overline{6}\frac{ft}{s} = 8.0\overline{6}\frac{ft}{s} \cdot \frac{1\,mi}{5280\,ft} \cdot \frac{60\,s}{1\,min} \cdot \frac{60\,min}{1\,hr} = \frac{(8.0\overline{6})\cdot 60\cdot 60}{5280}\frac{ft}{s} = 5.5\frac{ft}{s}$ It seems reasonable that the car will still be going very slow after just 1 second of acceleration, so 5.5 ft/s does seem reasonable.

Teaching Note: On the day when we look at the informal definition of limit in the book, or shortly thereafter when we start to learn ways of computing limits which are more efficient than trying to predict them from tables, we will go back and write the paragraph above using limit notation: $\lim_{t\to 1^+}\frac{s(t)-s(1)}{t-1} = 8.0\overline{6}\frac{ft}{s}$ and $\lim_{t\to 1^-}\frac{s(1)-s(t)}{1-t} = 8.0\overline{6}\frac{ft}{s}$ But since $\frac{s(1)-s(t)}{1-t}$ is equivalent to $\frac{s(t)-s(1)}{t-1}$, then $\lim_{t\to 1^-}\frac{s(1)-s(t)}{1-t} = 8.0\overline{6}\frac{ft}{s}$ can be rewritten as $\lim_{t\to 1^-}\frac{s(t)-s(1)}{t-1} = 8.0\overline{6}\frac{ft}{s}$. Thus since $\lim_{t\to 1^+}\frac{s(t)-s(1)}{t-1} = 8.0\overline{6}\frac{ft}{s}$ and $\lim_{t\to 1^-}\frac{s(t)-s(1)}{t-1} = 8.0\overline{6}\frac{ft}{s}$ then $\lim_{t\to 1}\frac{s(t)-s(1)}{t-1} = 8.0\overline{6}\frac{ft}{s}$

EXERCISE 4: Create a table similar to the one in Exercise 2 and use the table to predict the instantaneous speed of the car at t = 8 seconds.

Time Interval	Average speed in the time interval (in ft. per second)	Time Interval	Average speed in the time interval (in ft per second)
[8, 8.0001]	64.5337367	[7.9999, 8]	64.53293
[8, 8.00001]	64.5333736	[7.99999, 8]	64.533293
[8, 8.000001]	64.5333373	[7.999999, 8]	64.5333293
[8, 8.0000001]	64.5333337	7.9999999, 8]	64.5333329

As the time intervals in column 1 shrink down upon the instant in time t = 8 from the right, the average speeds in column 2 seem to be approaching $64.5\overline{3}\,\frac{ft}{s}$. As the time intervals in column 3 shrink down upon the instant in time t = 8 from the left, the average speeds in column 4 seem to be approaching $64.5\overline{3}\,\frac{ft}{s}$. Thus, the limit of the average speeds in time intervals shrinking down upon the instant in time t = 8 seems to be $64.5\overline{3}\,\frac{ft}{s}$. Therefore, I predict that the instantaneous speed at t = 8 (speedometer reading at t = 8) is $64.5\overline{3}\,\frac{ft}{s}$.

Teaching Note: Some students will say the limit approaches $64.5\overline{3}\,\frac{ft}{s}$ which is technically not correct. The average speeds approach $64.5\overline{3}\,\frac{ft}{s}$ so the limit is $64.5\overline{3}\,\frac{ft}{s}$. I try to correct this language. Note that Exercise 4 is as combination of Exercises 2 and 3, with the change from t = 1 to t = 8. Some students will not make the prediction because exercise 2 did not ask for the prediction so I mention this when assigning this problem.

EXERCISE 5: It turns out that the following is the speed function for the motion in this problem. (Note that the speed and velocity are the same in this problem, since the motion is such that the mile (or foot) markers are getting larger, that is the motion is in the positive direction. When motion is in the positive direction, velocity is positive, and speed and velocity are the same. When the motion is in the direction in which the mile (or foot) markers get smaller, we say the motion is in the negative direction, and the velocity is the negative of the speed.)

$v(t) = \begin{cases} \left(\frac{121}{15}\right)t & for\ 0 \le t \le 10 \\ \frac{242}{3} & for\ t > 10 \end{cases}$ where t is in ft/sec and t is in seconds.

Compute v(1) and v(8) using the function above. Does this agree with your previous predictions?

$v(1) = \left(\frac{121}{15}\right) \cdot 1 = \frac{121}{15} = 8.0\overline{6}\ \frac{ft}{s}$. Yes, this agrees with the prediction in Exercise 2.

$v(8) = \left(\frac{121}{15}\right) \cdot 8 = \frac{968}{15} = 64.5\overline{3}\ \frac{ft}{s}$. Yes, this agrees with the prediction in Exercise 4.