PHYS 3160 HOMEWORK ASSIGNMENT 01

DUE DATE JANUARY 03, 2020

Instructor: Dr. Daniel Erenso Name: ______________________________

Mandatory problems: 1 & 2

Student signature:____

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1. Find and solve the Euler-Lagrange equation that make the following integrals stationary.

(a)

$$I = \int_{t_1}^{t_2} \sqrt{t}\sqrt{1+\dot{q}^2}dt, \tag{1}$$

where t is the independent variable and $\dot{q} = \frac{dq}{dt}$.

(b)

$$I = \int_{y_1}^{y_2} \left(x^2 + x'^2\right) dy \tag{2}$$

here y is the independent variable and $x' = \frac{dx}{dy}$.

(b)

$$I = \int_{\theta_1}^{\theta_2} \theta\left(\sqrt{1-r'^2}\right) d\theta, \tag{3}$$

θ is the independent variable and $r' = \frac{dr}{d\theta}$.

Solution:

(a) For the integral the Euler-Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) - \frac{dL}{dq} = 0 \tag{4}$$

where

$$L(t, q, \dot{q}) = \sqrt{t}\sqrt{1+q'^2}, \tag{5}$$

which leads to

$$\frac{dL}{dq} = 0, \frac{dL}{dq'} = \frac{\sqrt{t}\dot{q}}{\sqrt{1+\dot{q}^2}}. \tag{6}$$

Then the Euler-Lagrange equation becomes

$$\frac{d}{dt}\left[\frac{\sqrt{t}\dot{q}}{\sqrt{1+\dot{q}^2}}\right] = 0 \Rightarrow \frac{\sqrt{t}\dot{q}}{\sqrt{1+\dot{q}^2}} = C, \tag{7}$$

where C is a constant. Simplifying the above expression, we find

$$\begin{aligned} t\dot{q}^2 &= C^2(1+\dot{q}^2) \Rightarrow \left(t - C^2\right)\dot{q}^2 = C^2 \Rightarrow \dot{q} = \frac{C}{\sqrt{t-C^2}} \\ \Rightarrow \quad \frac{dq}{dt} &= \frac{C}{\sqrt{t-C^2}} \Rightarrow \int_{q_1}^{q} dq = C\int_{t_1}^{t} \frac{dt}{\sqrt{t-C^2}} \end{aligned} \tag{8}$$

Let

$$\begin{aligned} u &= t - C^2 \Rightarrow du = dt, t = u + C^2 \\ q(u) - q_1 &= C\int_{u_1+C^2}^{u+C^2} \frac{du}{\sqrt{u}} = C\left(\sqrt{u+C^2} - \sqrt{u_1+C^2}\right) \end{aligned} \tag{9}$$

or

$$q(t) = C\left(\sqrt{t} - \sqrt{t_1}\right) + q_1 \Rightarrow q(t) = C_1\sqrt{t} + C_2. \tag{10}$$

(b) For this integral the independent variable is y and the Euler-Lagrange equation is given by

$$\frac{d}{dy}\left(\frac{dL}{dx'}\right) - \frac{dL}{dx} = 0 \tag{11}$$

where

$$L(x, y, y') = x^2 + x'^2. \tag{12}$$

Noting that

$$\frac{dL}{dx} = 2x, \frac{dL}{dx'} = 2x'. \tag{13}$$

the Euler-Lagrange equation becomes

$$\frac{d}{dy}(2x') - 2x = 0 \Rightarrow \frac{d^2x}{dy^2} - x = 0,$$

the solution of which is given by

$$x(y) = C_1 e^{-y} + C_2 e^{y}.$$

(c) For the integral

$$I = \int_{\theta_1}^{\theta_2} \theta\left(\sqrt{1 - r'^2}\right) d\theta, \tag{14}$$

θ is the independent variable and $r' = \frac{dr}{d\theta}$. Thus the Euler-Lagrange equation can be written as

$$\frac{d}{d\theta}\left(\frac{dL}{dr'}\right) - \frac{dL}{dr} = 0 \tag{15}$$

where

$$L(\theta, r, r') = \theta\left(\sqrt{1 - r'^2}\right). \tag{16}$$

Noting that

$$\frac{dL}{dr} = 0, \frac{dL}{dr'} = -\frac{\theta r'}{\sqrt{1 - r'^2}} = C. \tag{17}$$

where C is a constant. Simplifying the above expression, we find

$$\begin{aligned}
\theta^2 r'^2 &= C_1^2(1 - r'^2) \Rightarrow \left(\theta^2 + C^2\right) r'^2 = C_1^2 \Rightarrow r' = \frac{C_1}{\sqrt{\theta^2 + C^2}} \\
&\Rightarrow \frac{dr}{d\theta} = \frac{C_1}{\sqrt{\theta^2 + C_1^2}} \Rightarrow \int_{r_1}^{r} dr = C_1 \int_{\theta_1}^{\theta} \frac{d\theta}{\sqrt{\theta^2 + C_1^2}} \qquad (18) \\
&\Rightarrow r(\theta) = r_1 + C_1 \int_{\theta_1}^{\theta} \frac{d\theta}{\sqrt{\theta^2 + C_1^2}}. \qquad (19)
\end{aligned}$$

Introducing the transformation defined by

$$\theta = C_1 \sinh(u) \Rightarrow d\theta = C_1 \cosh(u)\, du \tag{20}$$

one can write the integral as

$$\int \frac{d\theta}{\sqrt{\theta^2 + C^2}} = \int \frac{C_1 \cosh(u)\, du}{\sqrt{C_1^2 \sinh^2(u) + C_1^2}} = \int du = u + C_2 \tag{21}$$

where we used the relation

$$\sinh^2(u) + 1 = \cosh^2(u) \tag{22}$$

and C_2 is a constant. Then noting that

$$u = \sinh^{-1}\left[\frac{\theta}{C_1}\right] \tag{23}$$

we can write

$$\int \frac{d\theta}{\sqrt{\theta^2 + C^2}} = \sinh^{-1}\left[\frac{\theta}{C_1}\right] + C_2 \tag{24}$$

so that Eq. (19) becomes

$$r(\theta) = r_1 + C_1\left[\sinh^{-1}\left[\frac{\theta}{C_1}\right] + C_2\right]_{\theta_1}^{\theta} = r_1 + C_1\left[\sinh^{-1}\left[\frac{\theta}{C_1}\right] + C_2\right] - C_1\left[\sinh^{-1}\left[\frac{\theta_1}{C_1}\right] + C_2\right]$$

$$\Rightarrow r(\theta) = C_1 \sinh^{-1}\left[\frac{\theta}{C_1}\right] + C_2' \tag{25}$$

where

$$C_2' = r_1 - C_1 \sinh^{-1}\left[\frac{\theta_1}{C_1}\right], \tag{26}$$

which is a constant.

2. From introductory physics you know that light travels with a speed, $c = 3 \times 10^8 m/s$, in vacuum. When it travels in a medium with a refractive index, $n > 1$ the light slows down and the speed, v is given by

$$v = \frac{c}{n}. \tag{27}$$

You also know that when light travels from one medium with refractive index $n_1$to another with refractive index, n_2, the light could bend towards or away from the normal depending on which refractive index is greater or less (Fig. 1). Using Calculus of variation show that,

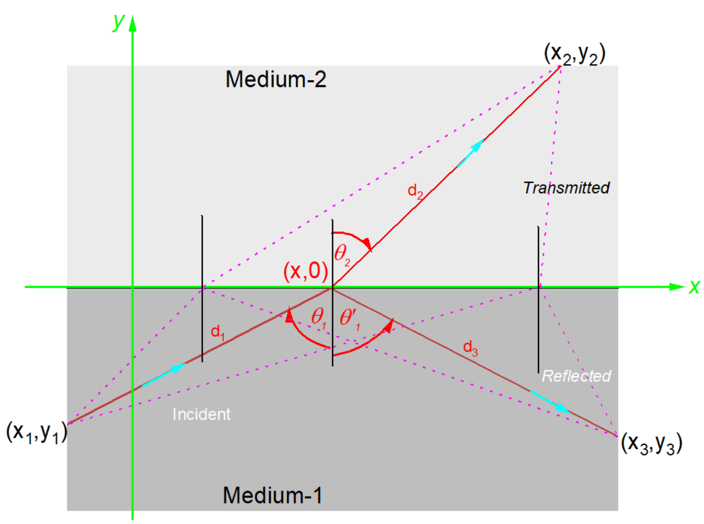

Figure 1: Different straight-line paths for the light traveling from point1 to point 2. The incident light partly gets reflected and partly gets refracted. The shortest paths are defined by the law of reflection and law of refraction.

(a) The angle of incidence, θ_1, and angle of transmission, θ_2, are related by the law of refraction (Snell's law)

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{28}$$

Hint:Minimize the time taken by the light

$$I = \int_1^2 dt, \tag{29}$$

Note that dt is an infinitessimal time for the light to travel an infinitessimal distance ds along the path of the light.

(b) The angle of incidence, θ_1, and the angle of reflection, θ_1',are related by the low of reflection

$$\theta_1 = \theta_1'. \tag{30}$$

Hint:Minimize the time taken by the light

$$I = \int_1^3 dt. \tag{31}$$

N.B. In either case the light travels in a straight line.

Solution:

(a) For the trajectory of the light, the total time to propagate from point 1 with (x_1, y_1) to point 2 (x_2, y_2) must be minimum. As shown in the figure, the light has traveled a distance d_1 and d_2 along a straight line in medium 1 and medium 2, respectively. Then the time taken by the light in the two media can be determined from the speed of light

$$v_1 = \frac{c}{n_1} \Rightarrow \frac{d_1}{t_1} = \frac{c}{n_1} \Rightarrow t_1 = \frac{n_1 d_1}{c}, \tag{32}$$

and

$$v_2 = \frac{c}{n_2} \Rightarrow \frac{d_2}{t_2} = \frac{c}{n_2} \Rightarrow t_2 = \frac{n_2 d_2}{c}. \tag{33}$$

so that the total time becomes

$$t = \frac{n_1 d_1 + n_2 d_2}{c} \tag{34}$$

Noting that

$$d_1 = \sqrt{(x - x_1)^2 + y_1^2}, d_2 = \sqrt{(x_2 - x)^2 + y_2^2} \tag{35}$$

one can write

$$t = \frac{1}{c}\left[n_1\sqrt{(x - x_1)^2 + y_1^2} + n_2\sqrt{(x_2 - x)^2 + y_2^2}\right]. \tag{36}$$

For different paths shown by the pink dotted lines, what varies is x.For the right path of the light, the time must be minimum. That means

$$\frac{dt}{dx} = 0 \tag{37}$$

so that

$$\frac{1}{c}\left[n_1\frac{x - x_1}{\sqrt{(x - x_1)^2 + y_1^2}} - n_2\frac{x_2 - x}{\sqrt{(x_2 - x)^2 + y_2^2}}\right] = 0. \tag{38}$$

From the figure shown we have

$$\sin\theta_1 = \frac{x - x_1}{\sqrt{(x - x_1)^2 + y_1^2}}, \sin\theta_2 = \frac{x_2 - x}{\sqrt{(x_2 - x)^2 + y_2^2}} \tag{39}$$

so that one can easily find

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{40}$$

which is Snell's law.

(b) Following a similar procedure for the reflected light one can write for the time to go from point 1 to point 3

$$t = \frac{1}{c}\left[n_1\sqrt{(x - x_1)^2 + y_1^2} + n_1\sqrt{(x_3 - x)^2 + y_3^2}\right]. \tag{41}$$

For different paths shown by the pink dotted lines, what varies is x.For the right path of the light, the time must be minimum. That means

$$\frac{dt}{dx} = 0 \tag{42}$$

so that

$$\frac{n_1}{c}\left[\frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}}-\frac{x_3-x}{\sqrt{(x_3-x)^2+y_3^2}}\right]=0. \tag{43}$$

From the figure shown we have

$$\sin\theta_1=\frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}},\sin\theta_1'=\frac{x_3-x}{\sqrt{x_B^2+(-y_B)^2}} \tag{44}$$

so that one can easily find

$$\sin\theta_1=\sin\theta_1'\Rightarrow\theta_1=\theta_1' \tag{45}$$

which is the low of reflection.

3. *This is a sort of reading assignment in my note or in the text.* Starting from

$$I(\epsilon)=\int_{x_1}^{x_2}F(x,Y,Y')\,dx,$$

where

$$Y(x)=y(x)+\epsilon\eta(x),$$

derive the *Euler-Lagrange Equation.*

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0,$$

by applying calculus of variation.

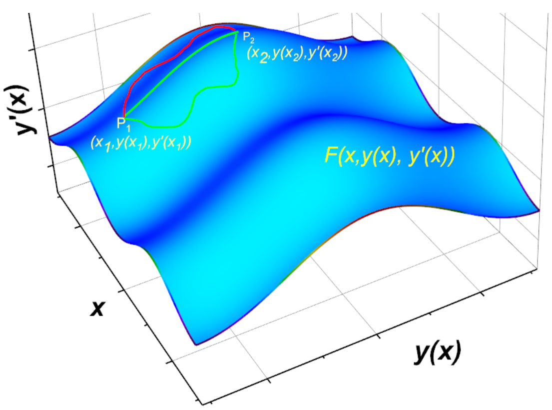

Figure 2: A surface defined by the function $F(x,y(x),y'(x)=\frac{dy}{dx})$

Solution: *The Euler-Lagrange Equation is derived by applying the calculus of variation. In general, in the problem that we want to solve applying the calculus of variation, we know the coordinates of two different points* $(x_1,y(x_1),y'(x_1))$ and $(x_2,y(x_2),y'(x_2))$ *on the surface defined by* $F=F(x,y(x),y'(x))$. From the infinitely many trajectories that can connect these two points, there is only one trajectory on this surface that is the shortest (the Geodesic). Finding the Geodesic is the general problem that can be solved applying the calculus of variation. The surface is defined by the function $F(x,y(x),y'(x))$. The distance between these two points determined by evaluating the integral

$$I=\int_{x_1}^{x_2}F(x,y(x),y'(x))\,dx. \tag{46}$$

To determine the equation that the function F is governed by so that we find the shortest length joining the two points, let the function for any path connecting the two points be $Y(x)$. From these infinite number of functions there is only one function that gives the minimum distance between the two points. If this function is $y(x)$, then we may write $Y(x)$ in terms of $y(x)$ as

$$Y(x,\epsilon) = y(x) + \epsilon\eta(x), \tag{47}$$

where $\eta(x)$ is an arbitrary function which must satisfy the condition

$$\eta(x_1) = \eta(x_2) = 0 \tag{48}$$

so that at the two points ($x = x_1 = x_2$), we find

$$Y(x,\epsilon) = y(x). \tag{49}$$

We also have

$$\frac{dY(x,\epsilon)}{d\epsilon} = \eta(x) \Rightarrow \left.\frac{dY(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \eta(x), \tag{50}$$

and

$$\frac{dY}{dx} = \frac{dy}{dx} + \epsilon\frac{d\eta}{dx} \text{ or } Y'(x,\epsilon) = y'(x) + \epsilon\eta'(x) \Rightarrow \left.Y'(x,\epsilon)\right|_{\epsilon=0} = y'(x), \tag{51}$$

which gives

$$\frac{dY'(\epsilon)}{d\epsilon} = \frac{d}{d\epsilon}\left[y'(x) + \epsilon\eta'(x)\right] = \eta'(x) \Rightarrow \left.\frac{dY'(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \eta'(x). \tag{52}$$

For the Geodesic the integral

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y(x,\epsilon), Y'(x,\epsilon))\, dx, \tag{53}$$

must be stationary, that means

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2} \frac{d}{d\epsilon}\left[F(x, Y(x,\epsilon), Y'(x,\epsilon))\right]\Big|_{\epsilon=0} dx = 0. \tag{54}$$

Noting that

$$\begin{aligned}\frac{d}{d\epsilon}\left[F(x, Y(x,\epsilon), Y'(x,\epsilon))\right]\Big|_{\epsilon=0} &= \frac{\partial F}{\partial Y}\frac{dY(\epsilon)}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} \\ &= \left.\frac{\partial F}{\partial Y}\right|_{\epsilon=0}\left.\frac{dY(\epsilon)}{d\epsilon}\right|_{\epsilon=0} + \left.\frac{\partial F}{\partial Y'}\right|_{\epsilon=0}\left.\frac{dY'(\epsilon)}{d\epsilon}\right|_{\epsilon=0}\end{aligned} \tag{55}$$

and substituting

$$\begin{aligned}\left.Y(x,\epsilon)\right|_{\epsilon=0} &= y(x), \left.\frac{dY(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \eta(x), \quad \left.Y'(x,\epsilon)\right|_{\epsilon=0} = y'(x), \\ \left.\frac{dY'(\epsilon)}{d\epsilon}\right|_{\epsilon=0} &= \eta'(x),\end{aligned} \tag{56}$$

we find

$$\begin{aligned}\frac{d}{d\epsilon}\left[F(x, Y(x,\epsilon), Y'(x,\epsilon))\right]\Big|_{\epsilon=0} &= \left[\frac{\partial}{\partial y}F(x, y(x), y'(x))\right]\eta(x) \\ &+ \left[\frac{\partial}{\partial y'}F(x, y(x), y'(x))\right]\eta'(x),\end{aligned} \tag{57}$$

Then the integral for the Geodesic line becomes

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\left[\frac{\partial}{\partial y}F(x, y(x), y'(x))\right]\eta(x)\, dx + \tag{58}$$

$$+ \int_{x_1}^{x_2}\left[\frac{\partial}{\partial y'}F(x, y(x), y'(x))\right]\eta'(x)\, dx. \tag{59}$$

Using integration by parts the second integral can be rewritten as

$$\int_{x_1}^{x_2} \left[\frac{\partial}{\partial y'} F(x, y(x), y'(x)) \right] \eta'(x)\, dx = \eta(x) \frac{\partial}{\partial x} \left[\frac{\partial F}{\partial y'} \right] \Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{\partial}{\partial x} \left[\frac{\partial F}{\partial y'} \right] \eta(x)\, dx, \tag{60}$$

so that using

$$\eta(x_1) = \eta(x_2) = 0, \tag{61}$$

we find

$$\int_{x_1}^{x_2} \left[\frac{\partial}{\partial y'} F(x, y(x), y'(x)) \right] \eta'(x)\, dx = - \int_{x_1}^{x_2} \eta(x) \frac{\partial}{\partial x} \left[\frac{\partial F}{\partial y'} \right] \eta(x)\, dx. \tag{62}$$

Thus the stationary integral can be put in the form

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial}{\partial y} F(x, y(x), y'(x)) \right] \eta(x)\, dx + - \int_{x_1}^{x_2} \eta(x) \frac{\partial}{\partial x} \left[\frac{\partial F}{\partial y'} \right] \eta(x)\, dx = 0. \tag{63}$$

or

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial}{\partial x} \left[\frac{\partial F}{\partial y'} \right] - \frac{\partial F}{\partial y} \right] \eta(x)\, dx = 0. \tag{64}$$

There follow that

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0 \tag{65}$$

where $F = F(x, y(x), y'(x))$.

4. Re-do Example 9.2 (in my note) applying Euler-Lagrange Equation,

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0,$$

or

$$\frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x'} \right) - \frac{\partial F}{\partial x} = 0,$$

whichever is easier.

Solution: We begin from the integral

$$L = \int_{(1)}^{(2)} \sqrt{1 + \left(\frac{dy}{dx} \right)^2}\, dx \tag{66}$$

which can be put in the form

$$L = \int_{x_1}^{x_2} F(x, y(x), y'(x))\, dx \tag{67}$$

where

$$F(x, y(x), y'(x)) = \sqrt{1 + y'^2}.$$

The noting that

$$\frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

The Euler-Lagrange equation becomes

$$\frac{\partial}{\partial x} \left(\frac{y'}{\sqrt{1 + y'^2}} \right) = 0 \Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = c.$$

where c is a constant. Upon solving for y'

$$y' = \left[\frac{c^2}{1-c^2}\right]^{1/2} = m. \tag{68}$$

Note that we have introduced another constant in terms of the constant c. There follows that

$$\frac{dy}{dx} = m \Rightarrow y(x) = mx + b, \tag{69}$$

which is equation of a straight line.

5. Consider a surface of revolution generated by revolving a curve $y(x)$ about the x-axis. The curve is required to pass through fixed end points (x_1, y_1) and (x_2, y_2) as shown in Fig. 3. Find the curve $y(x)$ that gives the minimum area for the resulting surface using the *Euler-Lagrange Equation.*

Solution: The surface area generated as shown in Fig.**??** can be determined using the surface integral

$$A = \int_{p_1}^{p_2} 2\pi y(x)\, ds = \int_{y_1}^{y_2} 2\pi y(x) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_{y_1}^{y_2} F(y, x(y), x'(y)) dy.$$

where

$$F(y, x(y), x'(y)) = 2\pi y\sqrt{1 + x'^2}.$$

Noting that

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial x'} = \frac{2\pi y x'}{\sqrt{1 + x'^2}},$$

and using *Euler-Lagrange Equation,*

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0.$$

we find

$$\begin{aligned}
\frac{d}{dx}\left(\frac{yx'}{\sqrt{1+x'^2}}\right) &= 0 \Rightarrow \frac{yx'}{\sqrt{1+x'^2}} = c \Rightarrow \frac{y^2 x'^2}{1+x'^2} = c^2 \\
&\Rightarrow \frac{dx}{dy} = \frac{c}{\sqrt{y^2 - c^2}} \Rightarrow x(y_2) = \int_{y_1}^{y_2} \frac{cdu}{\sqrt{u^2 - c^2}}
\end{aligned}$$

Introducing the transformation defined by

$$y = c\cosh(\theta) \Rightarrow dy = c\sinh(\theta)\, d\theta, \sqrt{y^2 - c^2} = c\sqrt{\cosh^2(\theta) - 1} = c\sinh(\theta)$$

we find

$$\int \frac{cdy}{\sqrt{y^2 - c^2}} = \int \frac{c^2 \sinh(\theta)\, d\theta}{c\sinh(\theta)} = c\theta.$$

Noting that

$$\theta = \cosh^{-1}\left(\frac{y}{c}\right)$$

one finds

$$\begin{aligned}
x(y_2) &= \int_{y_1}^{y_2} \frac{cdu}{\sqrt{u^2 - c^2}} = c\cosh^{-1}\left(\frac{y}{c}\right)\Big|_{y_1}^{y_2} = c\cosh^{-1}\left(\frac{y_2}{c}\right) + c_1 \\
&\Rightarrow \frac{x(y_2) - c_1}{c} = \cosh^{-1}\left(\frac{y_2}{c}\right) \Rightarrow y_2(x_2) = c\cosh\left(\frac{x_2 - c_1}{c}\right).
\end{aligned}$$

or more generally

$$y(x) = c\cosh\left(\frac{x - c_1}{c}\right)$$

6. The speed of an electromagnetic wave in an atmosphere increases in proportion to the height, $v(y) = y/b$, where $b > 0$ is some parameter describing the speed of the electromagnetic wave. This shows that, for example, the speed becomes zero when the height is zero (i.e. $y = 0 \Rightarrow v = 0$) which simulates the condition at the surface of a black hole, called its *even horizon*, where the gravitational force is so strong that the velocity of light goes to zero, thus even trapping the light. Find the optical path in an atmosphere. *Hint: the optical path must be a shortest path and do not expect a straight line.*

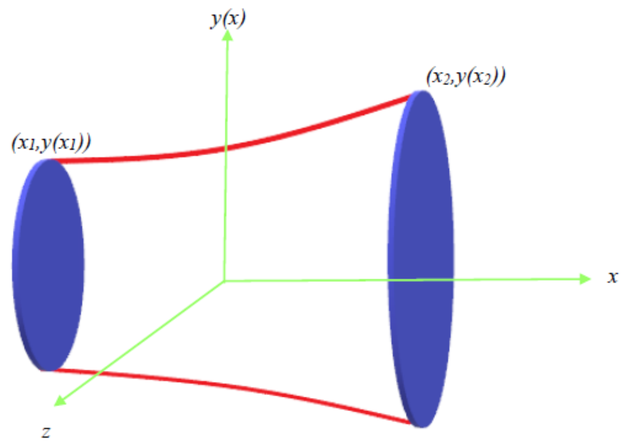

Figure 3: A curve rotated about the x-axis.

1.