PHYS 3160 HOMEWORK ASSIGNMENT 02

DUE DATE February 10, 2020

Instructor: Dr. Daniel Erenso Name: ____________________

Mandatory problems: 4, 6 (b) & (d)

Student signature:_______________________________________

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1. In Example 9.4 using *the Euler-Lagrange equation we had solved the brachystochrone problem, assuming the "material point" starts from rest, . We did show that in fact the shortest time if defined by an equation of an inverted cycloid*

$$x = \frac{1}{2c}(\theta - \sin(\theta)), y = -\frac{1}{2c}(1 - \cos(\theta)). \tag{1}$$

Show that when the material point starts with some initial velocity, v_0, still the shortest time path is defined by an inverted cycloid.

Solution: *If the particle is given an initial velocity $v_o \neq 0$, the path of minimum time must be minimum. That means we have to minimize the integral.*

$$I = \int_1^2 dt \tag{2}$$

From conservation of energy we have

$$\frac{1}{2}mv^2 - mgy = \frac{1}{2}mv_o^2, \tag{3}$$

where v_o is the initial velocity, so that

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_o^2 + mgy \Rightarrow v^2 = v_o^2 + 2gy \Rightarrow v = \sqrt{v_o^2 + 2gy}. \tag{4}$$

Recalling that

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \Rightarrow v = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\frac{dy}{dt}$$
$$\Rightarrow v = \sqrt{1 + x'^2}\frac{dy}{dt} \tag{5}$$

we may write

$$\sqrt{1 + x'^2}\frac{dy}{dt} = \sqrt{v_o^2 + 2gy} \Rightarrow I = \int_1^2 dt = \int_1^2 \frac{\sqrt{1 + x'^2}dy}{\sqrt{v_o^2 + 2gy}}, \tag{6}$$

where

$$x' = \frac{dx}{dy}. \tag{7}$$

Then the Euler-Lagrnage equation for the above integral becomes

$$\frac{d}{dy}\left(\frac{dL}{dx'}\right) - \frac{dL}{dx} = 0, \tag{8}$$

in which

$$L = \frac{\sqrt{1 + x'^2}}{\sqrt{v_o^2 + 2gy}} \tag{9}$$

Noting that

$$\frac{dL}{dx} = 0, \frac{dL}{dx'} = \frac{x'}{\sqrt{1 + x'^2}\sqrt{v_o^2 + 2gy}} \tag{10}$$

we Find

$$\frac{d}{dt}\left[\frac{x'}{\sqrt{1 + x'^2}\sqrt{v_o^2 + 2gy}}\right] = 0 \Rightarrow \frac{x'}{\sqrt{1 + x'^2}\sqrt{v_o^2 + 2gy}} = \sqrt{C} \tag{11}$$

where C is a constant solving for x'

$$x'^2 = C\left(1 + x'^2\right)\left(v_o^2 + 2gy\right) \Rightarrow x'^2\left[1 - C\left(v_o^2 + 2gy\right)\right] = C\left(v_o^2 + 2gy\right)$$
$$\Rightarrow x' = \frac{\sqrt{C\left(v_o^2 + 2gy\right)}}{\sqrt{1 - C\left(v_o^2 + 2gy\right)}} \Rightarrow \int_{y_1}^{y} \frac{\sqrt{C\left(v_o^2 + 2gy\right)}}{\sqrt{1 - C\left(v_o^2 + 2gy\right)}}dy = \int_{x_1}^{x} dx. \tag{12}$$

Assuming the initial positions are $(x_1, y_2) = (0,0)$, *we find*

$$x = \int_0^y \frac{\sqrt{C\left(v_o^2 + 2gy\right)}dy}{\sqrt{1 - C\left(v_o^2 + 2gy\right)}}. \tag{13}$$

Introducing the transformation defined by

$$C\left(v_o^2 + 2gy\right) = \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1 - \cos\theta\right) \tag{14}$$

we have

$$\begin{aligned} \sqrt{1 - C\left(v_o^2 + 2gy\right)} &= \sqrt{1 - \sin^2\left(\frac{\theta}{2}\right)} = \cos\left(\frac{\theta}{2}\right), \\ C2gdy = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta &\Rightarrow dy = \frac{1}{2cg}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta, \end{aligned} \tag{15}$$

so that the above integral becomes

$$\begin{aligned} x &= \int_{\theta_o}^{\theta} \frac{\sin\left(\frac{\theta}{2}\right)\frac{1}{2cg}\sin\left(\frac{\theta}{2}\right)\frac{\cos\theta}{2}d\theta}{\frac{\cos\theta}{2}} \Rightarrow x = \frac{1}{2cg}\int_{\theta_o}^{\theta}\sin\left(\frac{\theta}{2}\right)d\theta \\ &\Rightarrow \quad x = \frac{1}{2cg}\int_{\theta_o}^{\theta}\frac{1}{2}\left(1 - \cos\theta\right)d\theta \Rightarrow x = \frac{1}{4cg}\left[\theta - \sin\theta\right]\Big|_{\theta_o}^{\theta} \\ &\Rightarrow \quad x = x_o + \frac{1}{4cg}\left[\theta - \sin\theta\right] \Rightarrow x - x_o = \frac{1}{4cg}\left[\theta - \sin\theta\right], \end{aligned} \tag{16}$$

where

$$x_o = \frac{1}{4cg}\left[\theta_o - \sin\theta_o\right]. \tag{17}$$

and θ_o *is determined from*

$$C\left(v_o^2 + 2gy\right) = \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1 - \cos\theta\right) \tag{18}$$

for $y = y_1 = 0$, *which will be*

$$cv_o^2 = \frac{1}{2}\left(1 - \cos\theta_o\right) \Rightarrow \cos\theta_o = 1 - 2cv_o^2 \Rightarrow \theta_o = \cos^{-1}\left[1 - 2cv_o^2\right]$$

Solving for y, *we have*

$$\begin{aligned} 2cgy = -cv_o^2 + \frac{1}{2}\left(1 - \cos\theta\right) &\Rightarrow y = \frac{-cv_o^2}{2cg} + \frac{1}{4cg}\left(1 - \cos\theta\right) \\ &\Rightarrow y - y_o = \frac{1}{4cg}\left(1 - \cos\theta\right), \end{aligned} \tag{19}$$

where

$$y_o = \frac{cv_o^2}{2cg} \tag{20}$$

Therefore the path of minimum time is still a cycloid. The only difference is the center of the cycloid will not be zero if it is given an initial velocity. The center will be (x_o, y_o) *that depends on the initial velocity*

$$x - x_o = \frac{1}{4cg}\left[\theta - \sin\theta\right], y - y_o = \frac{1}{4cg}\left(1 - \cos\theta\right). \tag{21}$$

2. Consider the motion of a mass m moving under the influence of a central force (that is, a force acting only along the radial direction) given by

$$\vec{F} = -f(r)\hat{r} \tag{22}$$

for some function $f(r)$. Assume that the motion is confined to a plane and the position of the mass can be described using polar coordinates (r, φ)

$$\vec{r} = r\cos(\varphi)\,\hat{x} + r\sin(\varphi)\,\hat{y} \tag{23}$$

where $\varphi = 0$,as shown in Fig.1

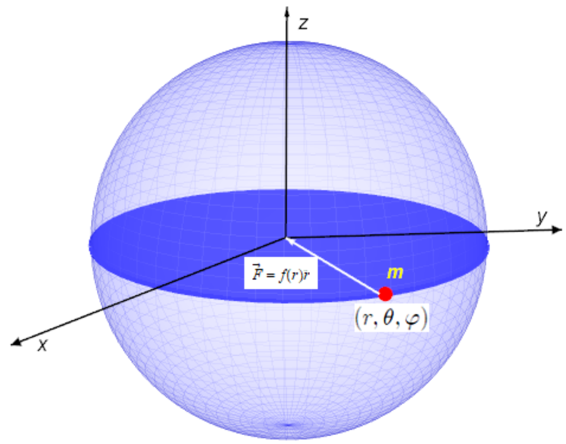

Figure 1: A mass m under a central force motion confined to the x-y plane.

(a) Find the Lagrangian

(b) Using Euler-Lagrange equation find the equation of motion for the mass for the radial and angular coordinates (i.e. r and φ). Show that one of these equations gives you the law of conservation of angular momentum, $\vec{L}$,

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = 0 \Rightarrow \vec{L} = I\vec{\omega} = \text{Constant}, \tag{24}$$

where $I = mr^2$ is the moment of inertia.

(c) For the case in which the radial distance is a constant

$$\dot{r} = \frac{dr}{dt} = 0 \tag{25}$$

you will find the equation of motion for a mass, m, moving in circle

$$mr\dot{\theta}^2 = -f(r) \Rightarrow \frac{m\left(r\dot{\theta}\right)^2}{r} = -f(r) \Rightarrow \frac{mv^2}{r} = -f(r), \tag{26}$$

Solution:

(a) *The kinetic energy*

$$T = \frac{1}{2}mv^2. \tag{27}$$

Using spherical coordinates $(r, \theta = \pi/2, \varphi)$ *the magnitude of the velocity can be expressed as*

$$\begin{aligned}
\vec{v} = \frac{d\vec{r}}{dt} &= \frac{d}{dt}\left[r\sin\left(\frac{\pi}{2}\right)\cos(\varphi)\,\hat{x} + r\sin\left(\frac{\pi}{2}\right)\sin(\varphi)\,\hat{y} + r\cos\left(\frac{\pi}{2}\right)\hat{z}\right] \\
\vec{v} &= \left[\dot{r}\cos(\varphi) - r\sin(\varphi)\right]\hat{x} + \left[\dot{r}\sin(\varphi) + r\cos(\varphi)\right]\hat{y} \\
&\Rightarrow v^2 = \dot{r}^2 + r^2\dot{\theta}^2
\end{aligned} \tag{28}$$

and the kinetic energy becomes

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right). \tag{29}$$

(b) *The potential energy is related to the central force by*

$$\vec{F} = -\nabla \cdot U(r) \tag{30}$$

where $U(r)$ is the potential energy. Since the force is a central force it is directed along the radial direction and it depends on r only. Therefore the potential energy can be expressed as

$$U(r) = \int f(r)\ dr. \tag{31}$$

Then the Lagrangian can be expressed as

$$L\left(t, r, \dot{r}, \theta, \dot{\theta}\right) = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - \int f(r)\ dr \tag{32}$$

Then using the Euler-Lagrange's equation

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{33}$$

we have

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{34}$$

so that using

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0, \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2 \dot{\theta}, \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - f(r), \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \tag{35}$$

we find

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = const \Rightarrow mr^2\dot{\theta} = cont \Rightarrow I\omega = cons. \text{ (Conservation of Ang. Mom.)}$$
$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \Rightarrow m\ddot{r} = mr\dot{\theta}^2 - f(r) \tag{36}$$

(c) For a circular motion where the radius is fixed, $\dot{r} = 0$, we find

$$mr\dot{\theta}^2 = -f(r) \Rightarrow \frac{m\left(r\dot{\theta}\right)^2}{r} = -f(r) \Rightarrow \frac{mv^2}{r} = -f(r), \tag{37}$$

which relates the centripetal acceleration, $a_c = v^2/r$, that you were introduced in introductory physics.

3. *A one-dimensional harmonic oscillator*: Consider a mass, m, attached to one end of a spring with spring constant, k,. Find the Lagrangian and the equation of motion for the mass using the Euler-Lagrange equation for the following two cases.

(a) The other end of the spring is attached to a wall as shown in Fig. and the mass is oscillating on a frictionless table.

(b) The other end of the spring is attached to a ceiling as shown in Fig.3 and the mass is oscillating in a vertical plane.

Solution:

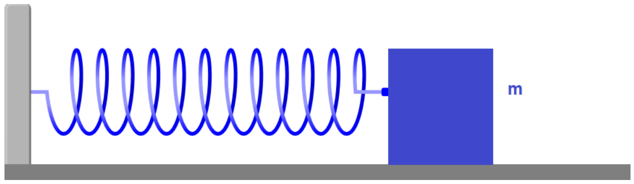

Figure 2: A harmonic oscillator in a horizontal plane.

Figure 3: A harmonic oscillator in a vertical plane.

(a) For a particle moving along the x axis with a potential energy

$$U = \frac{1}{2}kx^2 \tag{38}$$

the Lagrangian is given by

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{39}$$

Then the equation of motion is described by the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = 0 \Rightarrow \frac{d}{dt}(m\dot{x}) + kx = 0 \Rightarrow m\ddot{x} = -kx \Rightarrow F = ma = -kx \tag{40}$$

(b) In this case we have potential energy and also the motion is in the vertical plane. Thus one can write

$$U_{el} = \frac{1}{2}ky^2, U_g = mgy \tag{41}$$

and for the Lagrangian

$$L = T - U = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2 - mgy. \tag{42}$$

Then the equation of motion is described by the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{dL}{d\dot{y}}\right) - \frac{dL}{dy} = 0 \Rightarrow \frac{d}{dt}(m\dot{y}) + ky + mg = 0 \Rightarrow m\ddot{y} = -ky - mg \Rightarrow F = ma = ky + mg \tag{43}$$

4. Consider two masses, m_1 mass m_2, connected by a string of length, l, with negligible mass. The string passes through a hole at the center of a table . The mass m_1 is on the table and it can move on the table. The surface of the table is frictionless. The second mass, m_2, hanging from the other end of the string can move up or down on a vertical plane. (see Fig. 4.

(a) Using cylindrical coordinates (r, φ, z), find the Lagrangian

(b) Find the equation of motion for the two masses using Euler-Lagrange equation.

Solution:

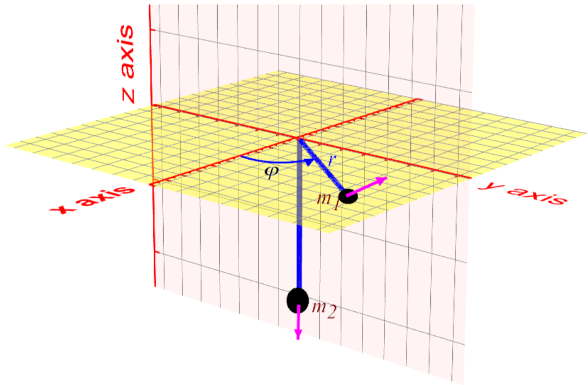

Figure 4: Two masses connected by a string of length l. In cylindrical coordinates the position for m_1 is $(r, \varphi, 0)$ and for m_2 is $(0, 0, -z_2)$. Note that $z_2 + r = l$.

(a) Using cylindrical coordinates the position of the mass on the table can be written as

$$\vec{r}_1 = r\cos(\theta)\,\hat{x} + r\sin(\theta)\,\hat{y} \tag{44}$$

and that of the second mass

$$\vec{r}_1 = -z\hat{z} \tag{45}$$

were we set the origin at the center of the table. Then Kinetic energy of the two masses can be expressed as

$$T = \frac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2\right] + \frac{1}{2}m\dot{z}^2. \tag{46}$$

The potential energy, assume the table is the ground level for gravitational potential energy, is given by

$$U = -mgz. \tag{47}$$

Noting that

$$r + |z| = l = \text{constant} \Rightarrow \dot{r} = -|\dot{z}| \tag{48}$$

one can write the Lagrangian as

$$L = T - U = \frac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2\right] + \frac{1}{2}m\dot{z}^2 + mgz = \frac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2\right] + \frac{1}{2}m\dot{r}^2 + mg(l-r)$$
$$\Rightarrow L = \frac{1}{2}m\left[2\dot{r}^2 + \left(r\dot{\theta}\right)^2\right] + mg(l-r). \tag{49}$$

(b) Using this Lagrangian, we have

$$\frac{dL}{d\dot{r}} = 2m\dot{r}, \frac{dL}{dr} = mr\dot{\theta}^2 - mg, \frac{dL}{d\dot{\theta}} = mr^2\dot{\theta}, \frac{dL}{d\theta} = 0 \tag{50}$$

so that for the equations of motion, given by

$$\frac{d}{dt}\left(\frac{dL}{d\dot{r}}\right) - \frac{dL}{dr} = 0, \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{dL}{d\theta} = 0, \tag{51}$$

one finds

$$\begin{aligned} \frac{d}{dt}(2m\dot{r}) - mr\dot{\theta}^2 - mg &= 0 \Rightarrow \ddot{r} = \frac{1}{2}\left(g - r\dot{\theta}^2\right), \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) &= 0 \Rightarrow mr^2\dot{\theta} = \text{constant} \end{aligned} \tag{52}$$

5. Find the magnitude of the vector pointing from point P to point Q, when these points are

(a) $P = (4, -1, 2, 7)$ and $Q = (2, 3, 1, 9)$.

(b) $P = (-1, 5, -3, 2, 4)$ and $Q = (2, 6, 2, 7, 6)$.

(c) Points described by the Minkowski space-time coordinates, $P = (x_1, y_1, z_1, ct_1)$ and $P = (x_2, y_2, z_2, ct_2)$ where c is the speed of light in free space. *You will see this in General relativity.*

Solution:

(a) The distance is given by

$$d = \sqrt{(2-4)^2 + (3--1)^2 + (1-2)^2 + (9-7)^2} = \sqrt{2^2 + 4^2 + 1^2 + 2^2} = 5 \tag{53}$$

(b) The distance is given by

$$d = \sqrt{(2--1)^2 + (6-5)^2 + (2--3)^2 + (7-2)^2 + (6-4)^2} = \sqrt{3^2 + 1^2 + 5^2 + 5^2 + 2^2} = 8 \tag{54}$$

(c)

6. For the matrices listed (a)-(d)

i. Find the eigenvalues and eigen vectors

ii. Construct the matrix C that diagonalizes each these matrices and determine its inverse matrix, C^{-1}

iii. Compute $C^{-1}MC$ for each matrices.(*this part may be done using Mathematica, in which case appropriate output must be provided*)

iv. Show that the matrices in (b) and (d) are Hermitian.

(a)

$$M = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \tag{55}$$

(b)

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{56}$$

These matrices are usually represented as σ_x, σ_y, and σ_z and are known as the Pauli Spin-1/2 matrices that you will see in quantum mechanics.

(c)

$$M = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{57}$$

(d)

$$M = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{58}$$

This matrix is also related to spin-matrices (but for spin-1 particles) and you will also see it in quantum mechanics.

Solution:

(a) From the eigenvalue equation we find that

$$\begin{vmatrix} 1-\epsilon & 3 \\ 2 & 2-\epsilon \end{vmatrix} = 0 \Rightarrow (1-\epsilon)\,(2-\epsilon) - 6 = 0 \Rightarrow \epsilon^2 - 3\epsilon - 4 = 0 \Rightarrow (\epsilon-4)(\epsilon+1) = 0$$
$$\Rightarrow \epsilon_1 = 4, \epsilon_2 = -1 \tag{59}$$

For the eigenvectors we must have:

$$\begin{bmatrix} 1-\epsilon_i & 3 \\ 2 & 2-\epsilon_i \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = 0 \tag{60}$$

Therefore for $\epsilon_1 = 4$, we have

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = 0 \Rightarrow 2A_1 - 2B_1 = 0 \Rightarrow A_1 = B_1 \tag{61}$$

Therefore

$$\epsilon_1 = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{62}$$

and after normalization

$$\epsilon_1^* \epsilon_1 = 1 \Rightarrow A_1 = \frac{1}{\sqrt{2}} \Rightarrow \epsilon_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{63}$$

For $\epsilon_2 = -1$, we have

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = 0 \Rightarrow 2A_2 + 3B_2 = 0 \Rightarrow B_2 = -\frac{2}{3}A_2 \tag{64}$$

and the eigenvector becomes

$$\epsilon_2 = A_2 \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix} \Rightarrow \epsilon_2 = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \tag{65}$$

after normalization. Then the matrix C, which must be constructed from the normalized eigen vectors, can be written as

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{13}} \end{bmatrix} \tag{66}$$

and the inverse matrix (which I determined using Mathematica)

$$C^{-1} = \begin{bmatrix} \frac{2\sqrt{2}}{5} & \frac{3\sqrt{2}}{\sqrt{13}} \\ -\frac{\sqrt{13}}{5} & \frac{\sqrt{13}}{5} \end{bmatrix} \tag{67}$$

One can easily see that

$$C^{-1}MC = \begin{bmatrix} \frac{2\sqrt{2}}{5} & \frac{3\sqrt{2}}{\sqrt{13}} \\ -\frac{\sqrt{13}}{5} & \frac{\sqrt{13}}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{13}} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \tag{68}$$

which is a diagonal matrix. The diagonal elements are the eigen values. These are checked using Mathematica

In[5]:= $\texttt{Simplify}\left[\texttt{Inverse}\left[\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{13}} \end{pmatrix}\right] \texttt{ // MatrixForm}\right]$

Out[5]//MatrixForm=

$$\begin{pmatrix} \frac{2\sqrt{2}}{5} & \frac{3\sqrt{2}}{5} \\ -\frac{\sqrt{13}}{5} & \frac{\sqrt{13}}{5} \end{pmatrix}$$

In[6]:= $\texttt{Simplify}\left[\texttt{Inverse}\left[\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{13}} \end{pmatrix}\right].\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}.\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{13}} \end{pmatrix}\right] \texttt{ // MatrixForm}$

Out[6]//MatrixForm=

$$\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

In[7]:= $\texttt{Eigensystem}\left[\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\right]$

Out[7]= {{4, -1}, {{1, 1}, {-3, 2}}}

(b) For the matrix the eigen values are

$$\det\begin{vmatrix} -\epsilon & 1 \\ 1 & -\epsilon \end{vmatrix} = 0 \Rightarrow \epsilon^2 - 1 = 0 \Rightarrow \epsilon_1 = 1 \text{ and } \epsilon_2 = -1. \tag{69}$$

The corresponding eigen vector, for $\epsilon_1 = 1$

$$\begin{bmatrix} -\epsilon_1 & 1 \\ 1 & -\epsilon_1 \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0 \Rightarrow A_1 = B_1 \tag{70}$$

Therefore the eigenvector can be expressed as

$$|\epsilon_1\rangle = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow |\epsilon_1\rangle = \frac{1}{\sqrt{2}} \begin{matrix} 1 \\ 1 \end{matrix}\,, \tag{71}$$

after normalizing it. Similarly for$\epsilon_2 = -1$

$$\begin{bmatrix} -\epsilon_2 & 1 \\ 1 & -\epsilon_2 \end{bmatrix}\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0 \Rightarrow B_1 = -A_1 \Rightarrow |\epsilon_2\rangle = \frac{1}{\sqrt{2}} \begin{matrix} 1 \\ -1 \end{matrix}\,. \tag{72}$$

Then the matrix C, which must be constructed from the normalized eigen vectors, can be written as

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{73}$$

and the inverse matrix (which I determined using Mathematica)

$$C^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{74}$$

One can easily see that

$$C^{-1}MC = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{75}$$

which is a diagonal matrix. The diagonal elements are the eigen values. These are checked using Mathematica

In[12]:= `Simplify[Inverse[`$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$`] // MatrixForm]`

Out[12]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

In[13]:= `Simplify[Inverse[`$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$`]`$\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$`] // MatrixForm`

Out[13]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In[14]:= `Eigensystem[`$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$`]`

Out[14]= `{{-1, 1}, {{-1, 1}, {1, 1}}}`

For the matrix

$$\det \begin{vmatrix} -\epsilon & -i \\ i & -\epsilon \end{vmatrix} = 0 \Rightarrow \epsilon^2 - 1 = 0 \Rightarrow \epsilon_1 = 1 \text{and } \epsilon_2 = -1. \tag{76}$$

The corresponding eigen vector, for $\epsilon_1 = 1$

$$\begin{aligned} \begin{bmatrix} -\epsilon_1 & -i \\ i & -\epsilon_1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} &= 0 \Rightarrow \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0 \Rightarrow A_1 = -iB_1 \text{ or } iA_1 = B_1 \\ |\epsilon_1\rangle &= A_1 \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow |\epsilon_1\rangle = \frac{1}{\sqrt{2}} \begin{matrix} 1 \\ i \end{matrix} . \end{aligned} \tag{77}$$

Similarly for $\epsilon_2 = -1$

$$\begin{bmatrix} -\epsilon_1 & -i \\ i & -\epsilon_1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$$

$$\Rightarrow A_1 = iB_1 \text{ or } -iA_1 = B_1 \Rightarrow |\epsilon_2\rangle = A_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} \Rightarrow |\epsilon_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{78}$$

Then for matrix, C , and its inverse, C^{-1}, we have

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}, C^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \tag{79}$$

so that

$$C^{-1}MC = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

```
In[16]:= Simplify[Inverse[( 1/√2   1/√2
                            i/√2  -i/√2 )] // MatrixForm]

Out[16]//MatrixForm=
( 1/√2  -i/√2
  1/√2   i/√2 )

In[17]:= Simplify[Inverse[( 1/√2   1/√2
                            i/√2  -i/√2 )].( 0  -i
                                             i   0 ).( 1/√2   1/√2
                                                       i/√2  -i/√2 )] // MatrixForm

Out[17]//MatrixForm=
( 1   0
  0  -1 )

In[18]:= Eigensystem[( 0  -i
                       i   0 )]

Out[18]= {{-1, 1}, {{i, 1}, {-i, 1}}}
```

The matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{80}$$

is already a diagonal matrix.

(c) For the eigenvalues

$$\begin{vmatrix} 2-\epsilon & 3 & 0 \\ 3 & 2-\epsilon & 0 \\ 0 & 0 & 1-\epsilon \end{vmatrix} = 0 \Rightarrow (2-\epsilon) \begin{vmatrix} 2-\epsilon & 0 \\ 0 & 1-\epsilon \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 0 & 1-\epsilon \end{vmatrix} = 0 \Rightarrow (\epsilon+1)(5-\epsilon)(1-\epsilon) = 0$$
$$\Rightarrow \epsilon_1 = -1, \epsilon_2 = 5, \epsilon_3 = 1 \tag{81}$$

The corresponding eigenvector for $\epsilon_1 = -1$,

$$\begin{bmatrix} 2+1 & 3 & 0 \\ 3 & 2+1 & 0 \\ 0 & 0 & 1+1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = 0$$
$$3A_1 + 3B_1 = 0, 3A_1 + 3B_1 = 0, 2C_1 = 0 \Rightarrow C_1 = 0, B_1 = -A_1 \Rightarrow |\epsilon_1\rangle = A_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow |\epsilon_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \tag{82}$$

For $\epsilon_2 = 5$

$$\begin{bmatrix} 2-5 & 3 & 0 \\ 3 & 2-5 & 0 \\ 0 & 0 & 1-5 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = 0$$
$$-3A_2 + 3B_2 = 0, 3A_2 - 3B_2 = 0, -4C_2 = 0 \Rightarrow C_2 = 0, B_2 = A_2 \Rightarrow |\epsilon_2\rangle = A_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow |\epsilon_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{83}$$

For $\epsilon_3 = 1$, we have

$$\begin{bmatrix} 2-1 & 3 & 0 \\ 3 & 2-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = 0$$
$$A_3 + 3B_3 = 0, 3A_3 + B_3 = 0 \Rightarrow A_3 = -3B_3, 3A_3 = -B_3 \tag{84}$$

The above two equations can be true if, and only if: $A_3 = B_3 = 0$. We can choose any value for C_3 since the eigenvalue equation is found to be independent of C_3. Therefore, the eigenvector can be rewritten as

$$|\epsilon_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{85}$$

(d) For the matrix

$$M = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{86}$$

the eigenvalues are

$$\begin{vmatrix} -\epsilon & -i & 0 \\ i & -\epsilon & -i \\ 0 & i & -\epsilon \end{vmatrix} = 0 \Rightarrow -\epsilon \begin{vmatrix} -\epsilon & -i \\ i & -\epsilon \end{vmatrix} + i \begin{vmatrix} i & -i \\ 0 & -\epsilon \end{vmatrix} = 0 \Rightarrow -\epsilon\left(\epsilon^2 - 1\right) + \epsilon = 0$$

$$\Rightarrow \epsilon(\epsilon^2 - 2) = 0 \Rightarrow \epsilon_1 = 0, \epsilon_2 = \sqrt{2}, \epsilon_3 = -\sqrt{2} \tag{87}$$

The eigenvector for $\epsilon_1 = 0$

$$\begin{aligned} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} &= 0 \Rightarrow -iB_1 = 0, iA_1 - iC_1 = 0, iB_1 = 0 \Rightarrow C_1 = A_1, B_1 = 0 \\ \Rightarrow & \quad |\epsilon_1\rangle = A_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow |\epsilon_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{88}$$

For $\epsilon_2 = \sqrt{2}$

$$\begin{bmatrix} -\sqrt{2} & -i & 0 \\ i & -\sqrt{2} & -i \\ 0 & i & -\sqrt{2} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = 0 \Rightarrow -\sqrt{2}A_1 - iB_1 = 0, iA_1 - \sqrt{2}B_1 - iC_1 = 0, iB_1 - \sqrt{2}C_1 = 0$$

$$\Rightarrow B_1 = i\sqrt{2}A_1, iA_1 - \sqrt{2}i\sqrt{2}A_1 - iC_1 = 0 \Rightarrow -iA_1 - iC_1 = 0 \Rightarrow C_1 = -A_1,$$

$$|\epsilon_2\rangle = A_1 \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix} \Rightarrow |\epsilon_2\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}. \tag{89}$$

For $\epsilon_3 = -\sqrt{2}$, we have

$$\begin{bmatrix} \sqrt{2} & -i & 0 \\ i & \sqrt{2} & -i \\ 0 & i & \sqrt{2} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = 0 \Rightarrow \sqrt{2}A_1 - iB_1 = 0, iA_1 + \sqrt{2}B_1 - iC_1 = 0, iB_1 + \sqrt{2}C_1 = 0$$

$$B_1 = -i\sqrt{2}A_1, iA_1 - \sqrt{2}i\sqrt{2}A_1 - iC_1 = 0 \Rightarrow -iA_1 - iC_1 = 0 \Rightarrow C_1 = -A_1$$

$$|\epsilon_3\rangle = A_1 \begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix} \Rightarrow |\epsilon_3\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}. \tag{90}$$

Using these eigen vectors, one find

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{i\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, C^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} \tag{91}$$

which leads to

$$C^{-1}MC = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{i\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix} \tag{92}$$

This is the result from Mathematica.

In[20]:= `MatrixForm[{{0, -i, 0}, {i, 0, -i}, {0, i, 0}}]`

Out[20]//MatrixForm=

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

In[29]:= Simplify $\left[\text{Inverse}\left[\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{i\sqrt{2}}{2} & \frac{-i\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right] \text{ // MatrixForm}\right]$

Out[29]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

In[30]:= Simplify $\left[\text{Inverse}\left[\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{i\sqrt{2}}{2} & \frac{-i\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right].\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{i\sqrt{2}}{2} & \frac{-i\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right] \text{ // MatrixForm}$

Out[30]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

In[31]:= Eigensystem $\left[\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\right]$

Out[31]= $\left\{\left\{-\sqrt{2}, \sqrt{2}, 0\right\}, \left\{\left\{-1, i\sqrt{2}, 1\right\}, \left\{-1, -i\sqrt{2}, 1\right\}, \{1, 0, 1\}\right\}\right\}$