PHYS 3160 HOMEWORK ASSIGNMENT 03

DUE DATE FEBRUARY 17, 2020

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Mandatory problems: 1 & 3

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**1.** Consider a system consisting of two masses $m_1$ and $m_2$ connected by three three springs with spring constant $k_1, k_2$, and $k_2$ as shown in Fig. 1.The masses can slide on a horizontal, frictionless surface. The springs are at

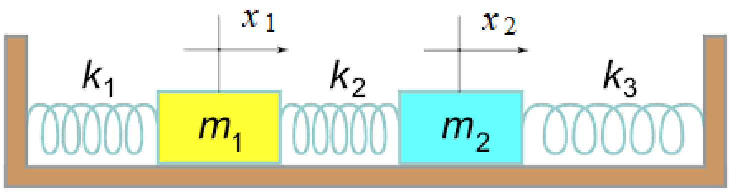


Figure 1: Two masses and three different springs.

their unstretched/uncompressed lengths when the masses are at its equilibrium positions. At $t = 0$, the masses are displaced from its equilibrium positions by the amounts $x_{10}$ and $x_{20}$ and released from rest.

(a) Find the kinetic energy, the potential energy, and the Lagrangian. **Using the Euler-Lagrange equation** derive the equations of motion for each masses and express the equations using matrices

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1}$$

(b) Let's assume that two atoms have nearly the same mass (i.e. $m_1 \simeq m_2 = m$) and ,

$$k_1 = 5k, k_2 = 2k, k_3 = 2k. \tag{2}$$

Using Similarity Transformation find the Eigenvalues and Eigenvectors for the matrix $M$.

(c) For the two masses find the displacements ($x_1(t)$ and $x_2(t)$) and speeds ($\dot{x}_1(t)$ and $\dot{x}_2(t)$)

(d) Find the propagator matrix.

(e) Describe the Normal Modes of Vibration of the atoms.

**Solution:**

(a) The kinetic energy can be expressed as

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2. \tag{3}$$

The elastic potential energy is given by

$$U = \frac{1}{2}\left[k_1x_1^2 + k_3x_2^2 + k_2(x_1 - x_2)^2\right] \tag{4}$$

Then the Lagrangian

$$L = T - U \tag{5}$$

becomes

$$L = \frac{1}{2}\left(m_1\dot{x}_1^2 + m_2\dot{x}_2^2\right) - \frac{1}{2}\left[k_1x_1^2 + k_3x_2^2 + k_2(x_1 - x_2)^2\right]. \tag{6}$$

The equations of motion, using Euler-Lagrange's equation,

$$\frac{d}{df}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = 0 \tag{7}$$

can be written as

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) \Rightarrow \ddot{x}_1 = -\left(\frac{k_2 + k_1}{m_1}\right)x_1 + \frac{k_2}{m_1}y, \\ m_2\ddot{x}_2 &= -k_3x_2 + k_2(x_1 - x_2) \Rightarrow \ddot{x}_2 = \frac{k_2}{m_2}x_1 - \left(\frac{k_2 + k_3}{m_2}\right)x_2. \end{aligned} \tag{8}$$

In a matrix from this can be put in the form

$$\begin{bmatrix} -\frac{k_2+k_1}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{9}$$

where

$$M = \begin{bmatrix} -\frac{k_2+k_1}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \end{bmatrix}. \tag{10}$$

(b) For the case $m_1 \simeq m_2 = m$ and $k_1 = 5k, k_2 = 2k, k_3 = 2k$, the matrix in part (a) becomes

$$M = \begin{bmatrix} -\frac{7k}{m} & \frac{2k}{m} \\ \frac{2k}{m} & -\frac{4k}{m} \end{bmatrix}.$$

The equation of motion

$$\frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

can be re-written as

$$\ddot{\vec{r}} = M\vec{r} \tag{12}$$

where

$$\vec{r} = x_1\hat{e}_1 + x_1\hat{e}_2 \Rightarrow \ddot{\vec{r}} = \ddot{x}_1\hat{e}_1 + \ddot{x}_1\hat{e}_2, \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{13}$$

We recall that for a matrix $M$ the similarity transformation is given by

$$D = T^{-1}MT, \tag{14}$$

where $T$ is a matrix whose columns are the eigenvectors and the matrix $D$ is a diagonal matrix where the elements are the eigenvalues to the Eigenvalue equation for matrix $M$. Suppose if we can find eigenvectors $\vec{R}$ such that

$$M\vec{R} = \lambda\vec{R}, \tag{15}$$

then the eigenvalue equation can be written as

$$\det \begin{vmatrix} -\frac{7k}{m} - \lambda & \frac{2k}{m} \\ \frac{2k}{m} & -\frac{4k}{m} - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 11\frac{k}{m}\lambda + 24\frac{k^2}{m^2} = 0, \tag{16}$$

and the eigenvalues become

$$\lambda_1 = -\frac{3k}{m}, \lambda_2. = -\frac{8k}{m}. \tag{17}$$

The corresponding eigenvectors are obtained using

$$\begin{bmatrix} -\frac{7k}{m} - \lambda_i & \frac{2k}{m} \\ \frac{2k}{m} & -\frac{4k}{m} - \lambda_i \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = 0. \tag{18}$$

For $\lambda_1 = -\frac{3k}{m}$, we find

$$\begin{bmatrix} -\frac{7k}{m} + \frac{3k}{m} & \frac{2k}{m} \\ \frac{2k}{m} & -\frac{4k}{m} + \frac{3k}{m} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} -\frac{4k}{m} & \frac{2k}{m} \\ \frac{2k}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$$
$$B_1 = 2A_1. \tag{19}$$

Similarly for $\lambda_2 = -\frac{8k}{m}$, one finds

$$\begin{bmatrix} -\frac{7k}{m} + \frac{8k}{m} & \frac{2k}{m} \\ \frac{2k}{m} & -\frac{4k}{m} + \frac{8k}{m} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \frac{k}{m} & \frac{2k}{m} \\ \frac{2k}{m} & \frac{4k}{m} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = 0$$
$$\Rightarrow A_2 = -2B_2 \tag{20}$$

Then the eigenvectors can be expressed as

$$|\lambda_1\rangle = A_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, |\lambda_2\rangle = B_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{21}$$

After normalization, we find

$$|\lambda_1\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, |\lambda_2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{22}$$

(c) Using the eigen vectors, the transformation matrix $T$ can then be written as

$$T = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}. \tag{23}$$

Using Mathematica I found the inverse matrix to be

$$T^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}. \tag{24}$$

Now recalling that

$$M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{25}$$

and noting that

$$TT^{-1} = T^{-1}T = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{26}$$

we can write

$$MTT^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{27}$$

Multiplying both sides from the left by $T^{-1}$, we have

$$T^{-1}MTT^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{d^2}{dt^2} T^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{28}$$

So that using

$$T^{-1}MT = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, T^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{29}$$

we can re-write Eq. (28) as

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \ddot{X} \\ \ddot{Y} \end{bmatrix} \Rightarrow \frac{d^2X}{dt^2} + \omega_1^2 X = 0, \frac{d^2Y}{dt^2} + \omega_2^2 Y = 0, \tag{30}$$

where

$$\omega_1^2 = -\lambda_1 = \frac{3k}{m}, \omega_2^2 = -\lambda_2 = \frac{8k}{m} \tag{31}$$

The solutions to the differential equations above are given by

$$X(t) = C\cos(\omega_1 t) + D\sin(\omega_1 t), Y(t) = E\cos(\omega_2 t) + F\sin(\omega_2 t) \tag{32}$$

Recalling that

$$T^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{33}$$

and using Eq. (26), we can write

$$TT^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T \begin{bmatrix} X \\ Y \end{bmatrix} \tag{34}$$

so that substituting the transformation matrix, $T$, we have

$$\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \frac{1}{\sqrt{5}}\left[\begin{array}{cc} 1 & -2 \\ 2 & 1 \end{array}\right]\left[\begin{array}{c} C\cos(\omega_1 t) + D\sin(\omega_1 t) \\ E\cos(\omega_2 t) + F\sin(\omega_2 t) \end{array}\right] \tag{35}$$

so that one finds for the position of the masses

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{5}}\left(C\cos(\omega_1 t) + D\sin(\omega_1 t)\right) - \frac{2}{\sqrt{5}}\left(E\cos(\omega_2 t) + F\sin(\omega_2 t)\right), \\ x_2(t) &= \frac{2}{\sqrt{5}}\left(C\cos(\omega_1 t) + D\sin(\omega_1 t)\right) + \frac{1}{\sqrt{5}}\left(E\cos(\omega_2 t) + F\sin(\omega_2 t)\right), \end{aligned} \tag{36}$$

and the corresponding speeds

$$\begin{aligned} \dot{x}_1(t) &= \frac{\omega_1}{\sqrt{5}}\left(-C\sin(\omega_1 t) + D\cos(\omega_1 t)\right) - \frac{2\omega_2}{\sqrt{5}}\left(-E\sin(\omega_2 t) + F\cos(\omega_2 t)\right), \\ \dot{x}_2(t) &= \frac{2\omega_1}{\sqrt{5}}\left(-C\sin(\omega_1 t) + D\cos(\omega_1 t)\right) + \frac{\omega_2}{\sqrt{5}}\left(-E\sin(\omega_2 t) + F\cos(\omega_2 t)\right). \end{aligned} \tag{37}$$

At the initial time, $t = 0$, we know that $\dot{x}_1(t) = \dot{x}_2(t) = 0$,

$$\begin{gathered} \dot{x}_1(t) = \frac{1}{\sqrt{5}}\left(\omega_1 D - 2\omega_2 F\right) = 0, \dot{x}_2(t) = \frac{1}{\sqrt{5}}\left(2\omega_1 D + \omega_2 F\right) = 0 \\ \Rightarrow D = F = 0 \end{gathered} \tag{38}$$

and we can re-write the positions as

$$x_1(t) = \frac{1}{\sqrt{5}}\left[C\cos(\omega_1 t) - 2E\cos(\omega_2 t)\right], x_2(t) = \frac{1}{\sqrt{5}}\left[2C\cos(\omega_1 t) + E\cos(\omega_2 t)\right]. \tag{39}$$

At the initial time the masses were displaced, $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$, which lead to

$$\frac{1}{\sqrt{5}}(C - 2E) = x_{10}, \frac{1}{\sqrt{5}}(2C + E) = x_{20} \Rightarrow C = \frac{x_{10} + 2x_{20}}{\sqrt{5}}, E = -\frac{2x_{10} - x_{20}}{\sqrt{5}}. \tag{40}$$

Therefore the position of the two masses are given by

$$\begin{aligned} x_1(t) &= \frac{1}{5}\left[(x_{10} + 2x_{20})\cos(\omega_1 t) + 2(2x_{10} - x_{20})\cos(\omega_2 t)\right], \\ x_2(t) &= \frac{1}{5}\left[2(x_{10} + 2x_{20})\cos(\omega_1 t) - (2x_{10} - x_{20})\cos(\omega_2 t)\right], \end{aligned} \tag{41}$$

and the velocity

$$\begin{aligned} \dot{x}_1(t) &= -\frac{1}{5}\left[(x_{10} + 2x_{20})\,\omega_1\sin(\omega_1 t) + 2\omega_2(2x_{10} - x_{20})\sin(\omega_2 t)\right], \\ \dot{x}_2(t) &= \frac{1}{5}\left[-2(x_{10} + 2x_{20})\,\omega_1\sin(\omega_1 t) + (2x_{10} - x_{20})\,\omega_2\sin(\omega_2 t)\right]. \end{aligned} \tag{42}$$

(d) The position of the two masses we found in part (c) and can be re-written as

$$\begin{aligned} x_1(t) &= \frac{1}{5}\left[(\cos(\omega_1 t) + 4\cos(\omega_2 t))\,x_{20} + 2(\cos(\omega_1 t) - \cos(\omega_2 t))\,x_{20}\right] \\ x_2(t) &= \frac{1}{5}\left[2(\cos(\omega_1 t) - \cos(\omega_2 t))\,x_{10} + (4\cos(\omega_1 t) + \cos(\omega_2 t))\,x_{20}\right]. \end{aligned} \tag{43}$$

so that in a matrix form, one finds

$$\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) = U(t)\left(\begin{array}{c} x_{20} \\ x_{20} \end{array}\right) \tag{44}$$

where

$$U(t) = \frac{1}{5}\left(\begin{array}{cc} \cos(\omega_1 t) + 4\cos(\omega_2 t) & 2(\cos(\omega_1 t) - \cos(\omega_2 t)) \\ 2(\cos(\omega_1 t) - \cos(\omega_2 t)) & 4\cos(\omega_1 t) + \cos(\omega_2 t) \end{array}\right) \tag{45}$$

is the evolution matrix.

(e) To describe the normal modes we assume first the initial state of the two masses is described by the first eigenvector. That means, we may write

$$\vec{r}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{46}$$

then

$$\vec{r} = U(t)\vec{r}_0$$

gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{5\sqrt{5}} \begin{pmatrix} \cos(\omega_1 t) + 4\cos(\omega_2 t) & 2(\cos(\omega_1 t) - \cos(\omega_2 t)) \\ 2(\cos(\omega_1 t) - \cos(\omega_2 t)) & 4\cos(\omega_1 t) + \cos(\omega_2 t) \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{47}$$

which leads to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{5\sqrt{5}} \begin{pmatrix} \cos(\omega_1 t) + 4\cos(\omega_2 t) + 4\cos(\omega_1 t) - 4\cos(\omega_2 t) \\ 2\cos(\omega_1 t) - 2\cos(\omega_2 t) + 8\cos(\omega_1 t) + 2\cos(\omega_2 t) \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \cos(\omega_1 t) \\ 2\cos(\omega_1 t) \end{pmatrix}$$
$$\Rightarrow x_1(t) = 2x_2(t) \tag{48}$$

The two masses oscillate with a frequency, $\omega_1$, in the same direction but with different amplitudes. On the other hand, if initially the state of the two masses is given by the second eigenvector

$$\vec{r}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{49}$$

then we find

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{5\sqrt{5}} \begin{pmatrix} \cos(\omega_1 t) + 4\cos(\omega_2 t) & 2(\cos(\omega_1 t) - \cos(\omega_2 t)) \\ 2(\cos(\omega_1 t) - \cos(\omega_2 t)) & 4\cos(\omega_1 t) + \cos(\omega_2 t) \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$
$$= \frac{1}{5\sqrt{5}} \begin{pmatrix} -2\cos(\omega_1 t) - 8\cos(\omega_2 t) + 2\cos(\omega_1 t) - 2\cos(\omega_2 t) \\ -4\cos(\omega_1 t) + 4\cos(\omega_2 t) + 4\cos(\omega_1 t) + \cos(\omega_2 t) \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2\cos(\omega_2 t) \\ \cos(\omega_2 t) \end{pmatrix}$$
$$\Rightarrow x_1(t) = -\frac{1}{2} x_2(t). \tag{50}$$

The two masses oscillate with a frequency $\omega_2$ out of phase but with different amplitudes.

**2.**

(a) Prove that

$$B(q, p) = B(p, q) \tag{51}$$

(b) Express the integrals

$$I_1 = \int_0^1 \frac{x^4}{\sqrt{1 - x^2}} dx, \quad I_2 = \int_0^\pi \sin^3(\theta)\cos(\theta) d\theta \tag{52}$$

as beta functions and then write each beta functions in terms of the Gamma functions using the relation we derived in *Example 6.2*,

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{53}$$

When possible use the Gamma function formulas such as

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, \quad \Gamma(p+1) = p\Gamma(p), \ \Gamma(1/2) = \sqrt{\pi} \tag{54}$$

to write an exact answer in terms of $\pi$, $\sqrt{2}$, etc.

(c) Applying the result in *Example 11.1* show that the integral

$$\int_{-\infty}^{\infty} e^{-y^2/a} dy = \sqrt{a\pi}, \tag{55}$$

for $a > 0$.

**Solution:**

(a) We recall

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx \tag{56}$$

So that

$$B(q,p)=\int_0^1 x^{q-1}(1-x)^{p-1}\,dx \tag{57}$$

Introducing a new variable

$$1-x=y\Rightarrow\left\{\begin{array}{c} x=1-y,-dy=dx,\\ x_1=0\Rightarrow y_1=1,\\ x_2=1\Rightarrow y_2=0\end{array}\right. \tag{58}$$

we may find

$$B(q,p)=\int_1^0 (1-y)^{q-1}\,y^{p-1}\,(-dy)=-\int_1^0 y^{p-1}(1-y)^{q-1}\,dy=\int_0^1 y^{p-1}(1-y)^{q-1}\,dy=B(p,q) \tag{59}$$

(b) The integral

$$I=\int_0^1\frac{x^4}{\sqrt{1-x^2}}dx \tag{60}$$

can be rewritten as

$$I=\int_0^1 x^4\left(1-x^2\right)^{-\frac{1}{2}}\,dx. \tag{61}$$

Introducing the transformation of variable defined by

$$x^2=y\Rightarrow\left\{\begin{array}{c} 2xdx=dy\Rightarrow dx=\frac{dy}{2x}=\frac{dy}{2\sqrt{y}},\\ x=0\Rightarrow y=0,\\ x=1\Rightarrow y=1,\end{array}\right. , \tag{62}$$

we find

$$I=\int_0^1 y^2(1-y)^{-\frac{1}{2}}\frac{dy}{2\sqrt{y}}\Rightarrow I=\frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}-1}\,dy. \tag{63}$$

Comparing this with the expression for the Beta function

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx \tag{64}$$

we note that the integral can be expressed as

$$I=\frac{1}{2}B\left(\frac{5}{2},\frac{1}{2}\right). \tag{65}$$

Using

$$B(p,q)=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \tag{66}$$

we find

$$\int_0^1\frac{x^4}{\sqrt{1-x^2}}dx=\frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}. \tag{67}$$

(c) Introducing the variable

$$t=y/\sqrt{a}\Rightarrow dy=\sqrt{a}dt \tag{68}$$

we have

$$\int_{-\infty}^{\infty}e^{-y^2/a}dy=\sqrt{a}\int_{-\infty}^{\infty}e^{-t^2}dt=2\sqrt{a}\int_0^{\infty}e^{-t^2}dt \tag{69}$$

Using the result from Example 11.1

$$\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2} \tag{70}$$

one can easily find

$$\int_{-\infty}^\infty e^{-y^2/a}dy = \sqrt{a\pi} \tag{71}$$

**3.** Using Stirling's formula evaluate

(a)

$$\lim_{n\to\infty}\left[\frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{n}\Gamma\left(n+1\right)}\right] \tag{72}$$

(b)

$$\lim_{n\to\infty}\left[\frac{(2n)!\sqrt{n}}{2^{2n}(n!)^2}\right] \tag{73}$$

**Solution:**

(a) Using the Stirling formula

$$\Gamma\left(p+1\right) \sim p^p e^{-p}\left(2\pi p\right)^{\frac{1}{2}} \tag{74}$$

we can write

$$\Gamma\left(n+\frac{3}{2}\right) = \Gamma\left(n+\frac{1}{2}+1\right) \Rightarrow \Gamma\left(n+\frac{3}{2}\right) \sim \left(n+\frac{1}{2}\right)^{\left(n+\frac{1}{2}\right)} e^{-\left(n+\frac{1}{2}\right)}\left(2\pi\right)^{\frac{1}{2}}\left(n+\frac{1}{2}\right)^{\frac{1}{2}}$$

$$\Rightarrow \Gamma\left(n+\frac{3}{2}\right) \sim \left(n+\frac{1}{2}\right)^{(n+1)} e^{-\left(n+\frac{1}{2}\right)}\sqrt{2\pi} \tag{75}$$

and

$$\Gamma\left(n+1\right) \sim n^n e^{-n}\left(2\pi\right)^{\frac{1}{2}} n^{\frac{1}{2}} \Rightarrow \Gamma\left(n+1\right) \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi} \Rightarrow \sqrt{n}\Gamma\left(n+1\right) \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi} \tag{76}$$

Therefore

$$\lim_{n\to\infty}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{n}\Gamma\left(n+1\right)} \sim \lim_{n\to\infty}\left[\frac{\left(n+\frac{1}{2}\right)^{(n+1)}e^{-\left(n+\frac{1}{2}\right)}\sqrt{2\pi}}{n^{n+1}e^{-n}\sqrt{2\pi}}\right] = e^{-\frac{1}{2}}\lim_{n\to\infty}\left[\left(1+\frac{1}{2n}\right)^{(n+1)}\right]. \tag{77}$$

Noting that

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{2n}\right)^{(n+1)}\right] = \lim_{n\to\infty}e^{\ln}\left[\left(1+\frac{1}{2n}\right)^{(n+1)}\right] = e^{\lim_{n\to\infty}\ln}\left[\left(1+\frac{1}{2n}\right)^{(n+1)}\right]$$

$$= e^{\lim_{n\to\infty}}\left[(n+1)\ln\left(1+\frac{1}{2n}\right)\right] = e^{\lim_{n\to\infty}}\left[\frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{1}{(n+1)}}\right] \Rightarrow^{\lim_{n\to\infty}}\left[\frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{1}{(n+1)}}\right] = \frac{0}{0}. \tag{78}$$

Hence, we can apply Le'Hospital rule which leads to

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{2n}\right)^{(n+1)}\right] = \lim_{n\to\infty}\frac{\frac{d}{dn}\left[\ln\left(1+\frac{1}{2n}\right)\right]}{\frac{d}{dn}\left[\frac{1}{(n+1)}\right]} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{2n}\right)\left(\frac{-1}{2n^2}\right)}{\frac{-1}{(n+1)^2}} = \lim_{n\to\infty}\frac{\frac{-1}{n(2n+1)}}{\frac{-1}{(n+1)^2}} = \lim_{n\to\infty}\frac{(n+1)^2}{n\left(2n+1\right)}$$

$$= \lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)^2}{n^2\left(2+\frac{1}{n}\right)} \Rightarrow \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{1}{n}\right)}\right] = \frac{1}{2} \Rightarrow \lim_{n\to\infty}\left[\left(1+\frac{1}{2n}\right)^{(n+1)}\right] = \lim_{n\to\infty}e^{\ln}\left[\left(1+\frac{1}{2n}\right)^{(n+1)}\right] = e^{\frac{1}{2}} \tag{79}$$

Therefore

$$\lim_{n\to\infty}\left[\frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{n}\Gamma\left(n+1\right)}\right] = e^{-\frac{1}{2}}\lim_{n\to\infty}\left[\frac{\left(n+\frac{1}{2}\right)^{n+1}}{n^{n+1}}\right] = e^{-\frac{1}{2}}e^{\lim_{n\to\infty}}\left[\frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{1}{(n+1)}}\right] = e^{-\frac{1}{2}}e^{\frac{1}{2}}$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{n}\Gamma\left(n+1\right)}\right] \simeq 1 \tag{80}$$

(b) We recall the Stirling's formula

$$n! \simeq n^n e^{-n}\sqrt{2\pi n} \tag{81}$$

so that for large $n$ we may write

$$\lim_{\longrightarrow\infty}\frac{(2n)!\sqrt{n}}{2^{2n}(n!)^2} = \frac{(2n)^{2n}e^{-2n}(2\pi 2nn)^{\frac{1}{2}}}{2^{2n}\left[n^n e^{-n}\sqrt{2\pi n}\right]^2} \Rightarrow \lim_{\longrightarrow\infty}\frac{(2n)!\sqrt{n}}{2^{2n}(n!)^2} \simeq \frac{\sqrt{\pi}}{\pi} = \frac{1}{\sqrt{\pi}} \tag{82}$$

**4.** The integral

$$\int_x^\infty u^{p-1}e^{-u}du = \Gamma(p,x) \tag{83}$$

is called an *incomplete Gamma function.* Note that for $x=0$, you find the Gamma function

$$\int_0^\infty u^{p-1}e^{-u}du = \Gamma(p). \tag{84}$$

By repeated integration find several terms of the asymptotic series for $\Gamma(p,x)$.

1. *NB: I found*

$$\begin{aligned}\Gamma(p,x) &= \int_x^\infty u^{p-1}e^{-u}du \\ &= x^{p-1}e^{-x}\left[1+(p-1)x^{-1}+(p-1)(p-2)x^{-2}+(p-1)(p-2)(p-3)x^{-3}...\right]\end{aligned} \tag{85}$$

**Solution:** Introducing the transformation defined by

$$u = f^{p-1} \Rightarrow du = (p-1)f^{p-2}df, dv = e^{-f} \Rightarrow v = -e^{-f} \tag{86}$$

and using integration by parts

$$\int udu = uv - \int vdv \tag{87}$$

the integral

$$I = \int_x^\infty f^{p-1}e^f df \tag{88}$$

can be expressed as

$$I = -f^{p-1}e^{-f}\big|_x^\infty + (p-1)\int_x^\infty f^{p-2}e^{-f}df = x^{p-1}e^{-x} + (p-1)\int_x^\infty f^{p-2}e^{-f}df \tag{89}$$

Following the same procedure we can show that

$$\int_x^\infty f^{p-2}e^{-f}df = x^{(p-2)}e^{-x} + (p-2)\int_x^\infty f^{p-3}e^{-f}df, \tag{90}$$

$$\begin{aligned}&\int_x^\infty f^{p-3}e^{-f}df = x^{(p-3)}e^{-x} + (p-3)\int_x^\infty f^{p-4}e^{-f}df, \\ \Rightarrow &\int_x^\infty f^{p-2}e^{-f}df = x^{(p-2)}e^{-x} + (p-2)x^{(p-3)}e^{-x} + (p-2)(p-3)\int_x^\infty f^{p-4}e^{-f}df\end{aligned} \tag{91}$$

Therefore

$$\begin{aligned}\int_x^\infty f^{p-1}e^{-f}df &= x^{(p-1)}e^{-x} + (p-1)x^{(p-2)}e^{-x} + (p-1)(p-2)x^{(p-3)}e^{-x} \\ &+ (p-1)(p-2)(p-3)\int_x^\infty f^{p-4}e^{-f}df...\end{aligned} \tag{92}$$

There follows that

$$\int_x^\infty f^{p-1}e^{-f}df = e^{-x}\left[x^{(p-1)} + (p-1)\,x^{p-2} + (p-1)\,(p-2)\,x^{p-3} + (p-1)\,(p-2)\,(p-3)\,x^{p-4}...\right]$$

$$\Rightarrow \int_x^\infty x^{p-1}e^{-f}df = x^{p-1}e^{-x}\left[1 + (p-1)\,x^{-1} + (p-1)\,(p-2)\,x^{-2} + (p-1)\,(p-2)\,(p-3)\,x^{-3}...\right] \tag{93}$$

**5.** Using the Gamma and Beta function formulas show that

$$\int_0^\infty \frac{dy}{(1+y)\sqrt{y}} = \pi \tag{94}$$

**6.**

(a) Prove that the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{95}$$

is an odd function.

(b) Show that

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right), \tag{96}$$

where

$$\operatorname{erf}\left(x/\sqrt{2}\right) = \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{2}} e^{-t^2}dt \tag{97}$$

is the error function.

**Solution:**

(a) Noting that

$$\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2}dt \tag{98}$$

and using the transformation defined by

$$t = -u \Rightarrow \left\{\begin{array}{c} dt = -du \\ t = 0 \Rightarrow u = 0 \\ t = -x \Rightarrow u = x \end{array}\right. \tag{99}$$

we find

$$\operatorname{erf}(-x) = -\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du = -\operatorname{erf}(x) \tag{100}$$

which shows that the error function is an odd function.

(a) Noting that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{-t^2/2}dt + \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}dt + \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt \tag{101}$$

The error function can also be expressed as

$$\frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) = \Phi(x) - \frac{1}{2} \Rightarrow \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right). \tag{102}$$