PHYS 3160 HOMEWORK ASSIGNMENT 04

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Mandatory problems: 2 & 5

Student signature:____

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P #	1	2	3	4	5	Score
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1. For the same pendulum in Example 11.5

(a) Use the Euler-Lagrange equation to find the equation of motion for the mass, m.

(b) The resulting equation is a none-linear differential equation. Show that this equation for small amplitude of oscillation gives a homogenous linear second order differential equation. By solving this equation show that the period of oscillation is given by same expression.

$$T = 2\pi\sqrt{\frac{l}{g}}, \tag{1}$$

Solution:

(a) For the mass, m, the kinetic energy is

$$KE = \frac{1}{2}m\left(\dot{\theta}l\right)^2 \tag{2}$$

and the gravitational potential energy is

$$PE = mgl(1 - \cos(\theta)). \tag{3}$$

Then using the Lagrangian

$$\mathcal{L} = KE - PE = \frac{1}{2}m\left(\dot{\theta}l\right)^2 - mgl(1 - \cos(\theta)), \tag{4}$$

The Euler-Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \tag{5}$$

becomes

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}m\left(\dot{\theta}l\right)^2 - mgl(1 - \cos(\theta))\right)\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2}m\left(\dot{\theta}l\right)^2 - mgl(1 - \cos(\theta))\right) = 0, \tag{6}$$

which gives

$$\frac{d}{dt}m\dot{\theta}l^2 + mgl\sin(\theta) = 0 \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0, \tag{7}$$

(b) For small angle θ, we have

$$\sin(\theta) \simeq \theta \tag{8}$$

so that the differential equation becomes

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \tag{9}$$

where

$$\omega = \sqrt{\frac{g}{l}}. \tag{10}$$

The solution to the differential equation is given by

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{11}$$

If the pendulum initially displaced an angle α and then released, we have

$$\theta(0) = \alpha \Rightarrow A = \alpha, \frac{d\theta(t)}{dt} = -A\omega\sin(\omega t) + \omega B\cos(\omega t) \Rightarrow \frac{d\theta(0)}{dt} = 0 \Rightarrow B = 0 \tag{12}$$

so that

$$\theta(t) = \alpha\cos(\omega t) = \alpha\cos(\omega t). \tag{13}$$

For one period, T, we must have

$$\theta(t) = \theta(t+T) \Rightarrow \alpha\cos(\omega t) = \alpha\cos(\omega(t+T)) \Rightarrow \cos(\omega t) = \cos(\omega t)\cos(\omega T) - \sin(\omega t)\sin(\omega T). \tag{14}$$

This equality holds only when

$$\cos(\omega T) = 1, \sin(\omega T) = 0 \Rightarrow \omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \tag{15}$$

2. Prove the relations

$$\delta(x) = \delta(-x), \delta(ax) = \frac{1}{a}\delta(x),$$

Solution: The Delta function is defined by one of its properties

$$\int_{-\infty}^{\infty} \sigma(x)\, dx = 1 \tag{16}$$

Let's consider the integral

$$I = \int_{-\infty}^{\infty} \sigma(-x)\, dx \tag{17}$$

so that introducing the transformation of variable defined by

$$y = -x \Rightarrow \begin{cases} x = -y \Rightarrow dx = -dy, \\ x = \infty \Rightarrow y = -\infty, \\ x = -\infty \Rightarrow y = \infty, \end{cases} \tag{18}$$

one can write

$$I = \int_{-\infty}^{\infty} \sigma(-x)\, dx = \int_{\infty}^{-\infty} \sigma(y)(-dy) = \int_{-\infty}^{\infty} \sigma(y)\, dy = 1 \tag{19}$$

according to the property of the Dirac delta function. Therefore

$$\int_{-\infty}^{\infty} \sigma(-x)\, dx = 1 = \int_{-\infty}^{\infty} \sigma(x)\, dx \Rightarrow \sigma(-x) = \sigma(x)$$

Let's consider the integral

$$I = \int_{-\infty}^{\infty} \delta(ax)\, dx$$

so that introducing the transformation of variable

$$y = ax \Rightarrow \begin{cases} x = \frac{y}{a} \Rightarrow dx = \frac{1}{a}dy, \\ x = \infty \Rightarrow y = \infty, \\ x = -\infty \Rightarrow y = -\infty, \end{cases} \tag{20}$$

one can write

$$I = \int_{-\infty}^{\infty} \delta(ax)\, dx = \int_{-\infty}^{\infty} \frac{1}{a}\delta(y)\, dy \tag{21}$$

but we know that the variable of integration is a dummy variable and one can rewrite this equation as

$$I = \int_{-\infty}^{\infty} \delta(ax)\, dx = \int_{-\infty}^{\infty} \frac{1}{a}\delta(x)\, dx \tag{22}$$

and there follows that

$$\delta(ax) = \frac{1}{a}\delta(x) \tag{23}$$

3. Show that

$$\int_{-\infty}^{\infty} \frac{d\delta(x)}{dx} f(x)\, dx = -\left.\frac{df(x)}{dx}\right|_{x=0} \tag{24}$$

Solution: Let's consider one of the properties of the Dirac Delta function

$$I = \int_{-\infty}^{\infty} \delta(x)\, g(x)\, dx = g(0)\,. \tag{25}$$

Suppose

$$g(x) = \frac{df(x)}{dx}$$

one can rewrite Eq. (25) as

$$\int_{-\infty}^{\infty} \delta(x) \frac{df(x)}{dx} dx = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{26}$$

Using integration by parts, we have

$$\delta(x)\, f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\delta(x)}{dx} f(x)\, dx = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{27}$$

so that using the property of the Dirac Delta function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \Rightarrow \delta(x)\, f(x)|_{-\infty}^{\infty} = 0$$

one finds

$$\int_{-\infty}^{\infty} \frac{d\delta(x)}{dx} f(x)\, dx = -\left.\frac{df(x)}{dx}\right|_{x=0}. \tag{28}$$

4. From introductory physics, the electric potential, $V(\vec{r})$, due to a point charge located at the origin $(0,0,0)$ (i.e. $r=0$) is given by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}. \tag{29}$$

Show that the volume charge density, $\rho(\vec{r})$, for this point charge can be expressed in terms of the Dirac delta function

$$\rho(\vec{r}) = \frac{dq}{d\tau} = q\sigma(\vec{r}) = q\sigma(x)\,\sigma(y)\,\sigma(z)\,, \tag{30}$$

where dq is an infinitessimal charge in an infinitesimal volume $d\tau$.

Solution: The electric potential, $dV(\vec{r})$ of an infinitessimal charge dq' in a volume $d\tau'$ as shown in Fig. can be expressed as

$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{dq'}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\epsilon_0}\frac{\rho(\vec{r}')\, d\tau'}{|\vec{r}-\vec{r}'|} \Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\vec{r}')\, d\tau'}{|\vec{r}-\vec{r}'|}, \tag{31}$$

Using spherical coordinates, we can write

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{\rho(\vec{r}')\, r'^2 \sin(\theta)\, dr' d\theta' d\varphi'}{|\vec{r}-\vec{r}'|} \tag{32}$$

This potential for a point charge becomes

$$\frac{1}{4\pi\epsilon_0}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{\rho(\vec{r}')\, r'^2 \sin(\theta)\, dr' d\theta' d\varphi'}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$\Rightarrow \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{1}{|\vec{r}-\vec{r}'|}\frac{\rho(\vec{r}')}{q} r'^2 \sin(\theta)\, dr' d\theta' d\varphi' = \frac{1}{r}. \tag{33}$$

From the property of the Dirac Delta function

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} f(\vec{r}')\,\sigma(\vec{r}' - \vec{r}_0)\,r'^2 \sin(\theta')\,dr'd\theta'd\varphi' = f(\vec{r}_0) \tag{34}$$

one can easily find

$$\begin{aligned} f(\vec{r}') &= \frac{1}{|\vec{r}-\vec{r}'|}, \sigma(\vec{r}'-\vec{r}_0) = \frac{\rho(\vec{r}')}{q} \\ &\Rightarrow\ f(\vec{r}_0) = \frac{1}{|\vec{r}-\vec{r}_0|} = \frac{1}{r} \Rightarrow \vec{r}_0 = 0 \end{aligned} \tag{35}$$

which leads to

$$\frac{\rho(\vec{r}')}{q} = \sigma(\vec{r}') \Rightarrow \rho(\vec{r}') = q\sigma(\vec{r}')\,. \tag{36}$$

where dq an infinitessimal charge in an infinitesimal volume $d\tau$.

5. The volume charge density, $\rho(\vec{r})$, of a point charge, q, placed at a point on the x-axis, $\vec{r}_0 = a\hat{x}$, can be expressed as

$$\rho(\vec{r}) = q\sigma(\vec{r}-\vec{r}_0)\,, \tag{37}$$

where $\sigma(\vec{r})$ is the Dirac Delta function. Show that the electric potential, $V(\vec{r})$, due to this point charge is given by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x-a)^2+y^2+z^2}}. \tag{38}$$

The electric potential for a volume charge distribution is given by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\vec{r}')\,d\tau'}{|\vec{r}-\vec{r}'|}, \tag{39}$$

where $\vec{r}'$ is the position of the infinitesimal charge $dq' = \rho(\vec{r}')\,d\tau'$, in an infinitesimal volume $d\tau'$, and $\rho(\vec{r}')$ is the charge density in the volume V.

Solution: Using the given charge density and the expression for the potential, one can write

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\iiint_V \frac{\sigma(\vec{r}'-\vec{r}_0)\,d\tau'}{|\vec{r}-\vec{r}'|}, \tag{40}$$

In Cartesian coordinates, we have

$$\begin{aligned} \vec{r}' &= x'\hat{x}+y'\hat{y}+z'\hat{z}, \vec{r} = x\hat{x}+y\hat{y}+z\hat{z}, \vec{r}_0 = a\hat{x} \\ &\Rightarrow\ |\vec{r}-\vec{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \end{aligned} \tag{41}$$

and

$$\sigma(\vec{r}'-\vec{r}_0) = \sigma(x'-x_0)\,\sigma(y'-y_0)\,\sigma(z'-z_0) = \sigma(x'-a)\,\sigma(y')\,\sigma(z')$$

so that

$$\begin{aligned} V(\vec{r}) &= \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sigma(x'-a)\,\sigma(y')\,\sigma(z')\,dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ &= \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\sigma(z')\,dz'\int_{-\infty}^{\infty}\sigma(y')\,dy'\int_{-\infty}^{\infty} f(x',y',z')\,\sigma(x'-a)\,dx', \end{aligned} \tag{42}$$

where

$$f\left(x',y',z'\right)=\frac{1}{\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2}}. \tag{43}$$

Now applying the property of the Dirac Delta function

$$\int_{-\infty}^{\infty} f\left(x\right)\sigma\left(x-a\right)dx=f\left(a\right) \tag{44}$$

one can easily see that

$$\begin{aligned}\int_{-\infty}^{\infty} f\left(x',y',z'\right)\sigma\left(x'\right)dx' &= \int_{-\infty}^{\infty} f\left(x',y',z'\right)\sigma\left(x'-a\right)dx' \\ &= f\left(a,y',z'\right)=\frac{1}{\sqrt{\left(x-a\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2}}.\end{aligned} \tag{45}$$

The electric potential becomes

$$V\left(\vec{r}\right)=\frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\sigma\left(z'-a\right)dz'\int_{-\infty}^{\infty} f\left(0,y',z'\right)\sigma\left(y'\right)dy'. \tag{46}$$

Once again using the property of the Dirac delta function, we have

$$\int_{-\infty}^{\infty} f\left(0,y',z'\right)\sigma\left(y'\right)dy'=\int_{-\infty}^{\infty} f\left(a,y',z'\right)\sigma\left(y'-0\right)dy'=f\left(a,0,z'\right)=\frac{1}{\sqrt{\left(x-a\right)^2+y^2+\left(z-z'\right)^2}}. \tag{47}$$

and the expression for potential reduces to

$$V\left(\vec{r}\right)=\frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty} f\left(a,0,z'\right)\sigma\left(z'\right)dz'. \tag{48}$$

One last time using the Dirac delta function property, we find for the potential

$$V\left(\vec{r}\right)=\frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty} f\left(0,0,z'\right)\sigma\left(z'\right)dz'=\frac{q}{4\pi\epsilon_0}f\left(0,0,0\right)\Rightarrow V\left(\vec{r}\right)=\frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\left(x-a\right)^2+y^2+z^2}}. \tag{49}$$

6. Show that

$$\delta\left[\left(x-x_1\right)\left(x-x_2\right)\right]=\frac{\delta\left(x-x_1\right)+\delta\left(x-x_2\right)}{\left|x_1-x_2\right|} \tag{50}$$

Solution: Introducing the transformation of variable

$$\begin{aligned}\left(x-x_1\right)\left(x-x_2\right)=y &\Rightarrow x=\frac{x_1+x_2+2\left[y+\frac{1}{4}\left(x_1-x_2\right)^2\right]^{1/2}}{2} \\ dy=dx\left(x-x_2\right)+dx\left(x-x_1\right)\Rightarrow dx &= \frac{dy}{2x-\left(x_2+x_1\right)}=\frac{dy}{2\left[y+\frac{1}{4}\left(x_1-x_2\right)^2\right]^{1/2}}\end{aligned} \tag{51}$$

we may write

$$\int_{-\infty}^{\infty}\delta\left[\left(x-x_1\right)\left(x-x_2\right)\right]dx=\int_{-\infty}^{\infty} f\left(y\right)\delta\left(y\right)dy \tag{52}$$

where

$$f\left(y\right)=\frac{1}{2\left[y+\frac{1}{4}\left(x_1-x_2\right)^2\right]^{1/2}}. \tag{53}$$

Using the property of the Delta function

$$\int_{-\infty}^{\infty} f\left(y\right)\sigma\left(y\right)dy=f\left(0\right) \tag{54}$$

we find

$$\int_{-\infty}^{\infty} \delta\left[\left(x-x_1\right)\left(x-x_2\right)\right]dx=\frac{1}{2\left[\frac{1}{4}\left(x_1-x_2\right)^2\right]^{1/2}}=\frac{1}{\left|x_1-x_2\right|} \tag{55}$$

Now lets consider the integral

$$I=\int_{-\infty}^{\infty}\left[\frac{\delta\left(x-x_1\right)+\delta\left(x-x_2\right)}{x_1-x_2}\right]dx \tag{56}$$

which we can be easily shown that

$$I=\int_{-\infty}^{\infty}\frac{\sigma\left(x-x_1\right)}{\left|x_1-x_2\right|}dx+\int_{-\infty}^{\infty}\frac{\sigma\left(x-x_2\right)}{\left|x_1-x_2\right|}dx=\frac{2}{\left|x_1-x_2\right|} \tag{57}$$