# PHYS 3160 HOMEWORK ASSIGNMENT 08 

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Instructor: Dr. Daniel Erenso
Name: $\qquad$

Mandatory problems: 3 \& 5
Student signature: $\qquad$

Comment: $\qquad$



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1. In theoretical Physics volume III we will be introduced to the electromagnetic wave equations. These equations are determined from Maxwell's equations in free space where there is no charge and current, given by

$$
\nabla \cdot \vec{E}=0, \nabla \cdot \vec{B}=0, \nabla \times \vec{E}=-\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B}=\frac{1}{c^{2}} \frac{\partial \vec{E}}{\partial t}
$$

which we will also derive in this volume Note that $c$ is the speed of light in vacuum. Using Maxwell's equation show that the electric field $(\vec{E})$ and the magnetic field $(\vec{B})$ satisfy the three dimensional wave equations

$$
\begin{equation*}
\nabla^{2} \vec{E}=\frac{1}{c^{2}} \frac{\partial^{2} \vec{E}}{\partial t^{2}}, \nabla^{2} \vec{B}=\frac{1}{c^{2}} \frac{\partial^{2} \vec{B}}{\partial t^{2}} \tag{1}
\end{equation*}
$$

Hint: take the curl of the curl of the electric field and apply the vector identity

$$
\begin{equation*}
\nabla \times(\nabla \times \vec{A})=\nabla(\nabla \cdot \vec{A})-\nabla^{2} \vec{A} \tag{2}
\end{equation*}
$$

2. Consider a string (for example a piano or violin or a guitar string) with linear mass density, $\lambda$, be stretched tightly by a tension force $T$ and its ends fastened to supports at $x=0$ and $x=l$. When the string is plucked (pulled aside a small distance and let go) and the string is vibrating (see Fig. 1), let represent its vertical displacement from its equilibrium position at a given point on the string $(x)$ and at a given time $(t)$ be $f(x, t)$. Consider an


Figure 1: The top is a snap shot of a tensioned string. The bottom is a snap shot of same string after a wave (a disturbance) is created on a strong. The incident and reflected wave at the ends form a standing wave.


Figure 2: The displacement of the wave from the equilibrium position, $f(z)$ vs $z$. Note that I did not show time dependence for the function $(f(x, t))$ in the graph as it represent a snapshot of the vibration.
infinitessimal segment of the string shown in Fig. 2. When the string is displaced from the equilibrium, the net transverse force on the segment between $x$ and $x+\Delta x$ can be expressed as

$$
\Delta F=T(\sin (\alpha)-\sin (\theta))
$$

We assume the equilibrium displacement $f(x, t)$ is always very small and that the slope at $x$ and $x+\Delta x$, given by

$$
\tan (\alpha)=\left.\frac{\partial f}{\partial x}\right|_{x}, \quad \tan (\theta)=\left.\frac{\partial f}{\partial x}\right|_{x+\Delta x}
$$

are very small and one can make the approximation

$$
\sin (\alpha) \simeq \tan (\alpha), \sin (\theta) \simeq \tan (\theta)
$$

Using this approximation and Newton's second law

$$
\begin{equation*}
\Delta F=m a=m \frac{\partial^{2} f}{\partial t^{2}} \tag{3}
\end{equation*}
$$

in the limit as $\Delta x \rightarrow 0$, show that the vibrating string obeys the wave equation

$$
\begin{equation*}
\frac{\partial^{2} f}{\partial x^{2}}=\frac{1}{v^{2}} \frac{\partial^{2} f}{\partial t^{2}} \tag{4}
\end{equation*}
$$

where $v$ is the speed of the wave

$$
\begin{equation*}
v=\sqrt{\frac{T}{\lambda}} \tag{5}
\end{equation*}
$$

Hint: the mass is given by

$$
\begin{equation*}
m=\lambda \Delta z \tag{6}
\end{equation*}
$$

note that

$$
\begin{equation*}
\frac{\partial^{2} f}{\partial x^{2}}=\lim _{\Delta z \rightarrow 0}\left[\frac{\left.\frac{\partial f}{\partial x}\right|_{x+\Delta x}-\left.\frac{\partial f}{\partial x}\right|_{x}}{\Delta x}\right] \tag{7}
\end{equation*}
$$

3. Using separation of variables

$$
\begin{equation*}
f(x, t)=X(x) T(t) \tag{8}
\end{equation*}
$$

in the wave equation for the vibrating string

$$
\begin{equation*}
\frac{\partial^{2} f}{\partial x^{2}}=\frac{1}{v^{2}} \frac{\partial^{2} f}{\partial t^{2}} \tag{9}
\end{equation*}
$$

(a) show that

$$
\begin{equation*}
\frac{d^{2} X}{d x^{2}}+k^{2} X=0, \frac{d^{2} T}{d t^{2}}+k^{2} v^{2} T=0 \tag{10}
\end{equation*}
$$

(b) Write the solutions to the differential equations in part (a).
(c) The string is fastened at $x=0$ and $x=l$. This means the displacement from the equilibrium position at these two points at any time $t$ is zero, which means, conditions

$$
\begin{equation*}
f(0, t)=0, f(l, t)=0 \tag{11}
\end{equation*}
$$

Furthermore one begins the vibration by plucking the string a small distance, $h$, for example near center of the string as shown Fig. 3 This means at $t=o$, the shape of the string is defined by some function $F_{0}(x)=f(x, 0)$ and none of the points are vibrating at this instant,

$$
\begin{equation*}
\left.\frac{d f(x, t)}{d t}\right|_{t=0}=0 \tag{12}
\end{equation*}
$$

Using these boundary conditions and the the solutions you determined in part (b), show that the solution to the wave equation is given by

$$
\begin{equation*}
f(x, t)=\sum_{n=1}^{\infty} b_{n} \sin \left(\frac{n \pi x}{l}\right) \cos \left(\frac{n \pi v t}{l}\right) \tag{13}
\end{equation*}
$$

Note: wave speed, $v$, is given by

$$
\begin{equation*}
v=\lambda f \tag{14}
\end{equation*}
$$

where $\lambda$ is the wavelength, and $f$ is the frequency of the wave and also the wave number, $k$, is defined as

$$
\begin{equation*}
k=\frac{2 \pi}{\lambda}=\frac{2 \pi f}{\lambda f}=\frac{\omega}{v} \tag{15}
\end{equation*}
$$

where $\omega=2 \pi f$ is the angular frequency.
(d) The shape of the string when it is plucked at the center at $t=0$ (see Fig. 3) is given by

$$
F_{0}(x)=f(x, 0)=\left\{\begin{array}{cc}
\frac{2 h}{l} x & 0<x<l / 2  \tag{16}\\
\frac{2 h}{l}(l-x) & l / 2<x<l
\end{array}\right.
$$

show that constants $b_{n}$ in Eq. (13)


Figure 3: The shape of the string (the displacement from the equilibrium position) when it is plucked at the center at the intial time (i.e. $f(x, 0)$ vs $x$ ). Note that here $f(x)$ is $F_{0}(x)=f(x, 0)$

$$
b_{n}=\frac{8 h}{n^{2} \pi^{2}}\left[2 \sin \left(\frac{n \pi}{2}\right) \sin ^{2}\left(\frac{n \pi}{4}\right)\right]
$$

4. In problem 3 we assume the string is plucked at the initial time $(t=0)$ at the center $(x=0.5 l)$. Let's consider the case where you plucked the string at $x=0.25 l$ near the lower end of the string (like you do in a Guitar).
(a) Write the function that defines the shape of the string at $t=0$.
(b) Find the function describing the displacement of the string from the equilibrium position at any time $t, f(x, t)$.
5. Find the steady-state temperature distribution in a plate with the boundary temperature $u=30^{\circ}$ for $x=0$ and $y=3 ; u=20^{\circ}$ for $x=5$ and $y=0$. For two dimensional heat flow the temperature, $u(x, y, t)$, satisfy the partial differential equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}-\frac{1}{\alpha^{2}}\left[\frac{\partial^{2} u(x, y, t)}{\partial x^{2}}+\frac{\partial^{2} u(x, y, t)}{\partial y^{2}}\right]=0 \tag{17}
\end{equation*}
$$

and at steady-state

$$
\frac{\partial u(x, y, t)}{\partial t}=0 \Rightarrow \frac{\partial^{2} u(x, y)}{\partial x^{2}}+\frac{\partial^{2} u(x, y)}{\partial y^{2}}=0
$$

Also for a temperature function defined by

$$
u^{\prime}(x, y)=u(x, y)-20^{\circ}
$$

at steady-state the partial differential equation is still

$$
\frac{\partial^{2} u^{\prime}(x, y)}{\partial x^{2}}+\frac{\partial^{2} u^{\prime}(x, y)}{\partial y^{2}}=0
$$

6. A bar of length $l$ siting on the $x$ axis with on end at the origin is initially $(t=0)$ at a temperature $u(x, 0)=0^{\circ}$. From $t=0$ on the ends are held at $20^{\circ}$ (i.e. $u(0, t)=u(l, t)=20^{\circ}$ ), find the temperature, $u(x, 0)$, for $t>0$.
