PHYS 4330 ELECTRICITY & MAGNETISM II
REVIEW HOMEWORK ASSIGNMENT 00
DUE DATE: February 04, 2020

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Mandatory problems: 4 (a) & (b), 5

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Problem 1-4 Electrostatic: A ring dipole: Consider a ring with radius R siting on the x-z plane centered about the origin as shown in Fig. 1. Suppose the angle between a vector, $\vec{r}'$ describing a point on the ring and the z-axis is φ' and the angle between a vector $\vec{r}$ describing a point in space and the y-axis is θ. The projection of the vector $\vec{r}$ on the x-z plane subtends an angle φ from the z-axis. The ring carries a positive charge q on one side and a negative charge $-q$ on the other side. These charges are uniformly distributed on both sides.

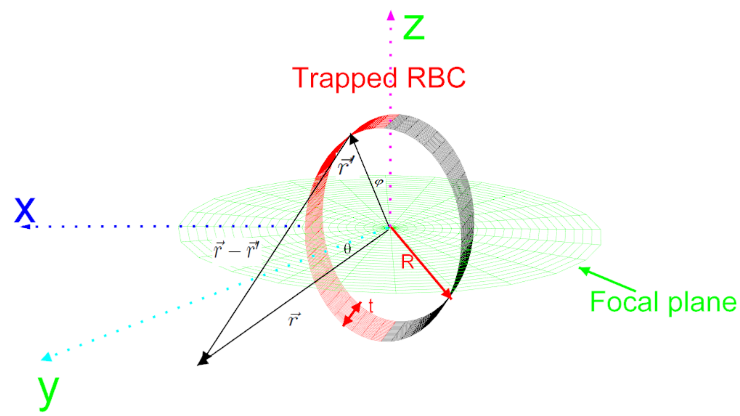


Figure 1: A simplified model for a trapped red blood cell. The cell is modeled a linear dielectric disk. When it is placed in an electric field, $\vec{E}(\vec{r})$, there will be a polarization. We assume the bound charges produced the the two sides of the cell are uniformly distributed.

1. For the set-up shown in the Fig. Noting that the Cartesian coordinate for the vector, $\vec{r}$,

$$x = r\sin(\theta)\sin(\varphi), y = r\cos(\theta), z = r\sin(\theta)\cos(\varphi) \tag{1}$$

and for the vector, $\vec{r}'$,

$$x' = R\sin(\varphi'), y' = 0, z' = R\cos(\varphi') \tag{2}$$

show that

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')} \tag{3}$$

Solution: Noting that

$$\begin{aligned}\vec{r} &= r\sin(\theta)\sin(\varphi)\hat{x} + r\cos(\theta)\hat{y} + r\sin(\theta)\cos(\varphi)\hat{z}\\ \vec{r}' &= R\sin(\varphi')\hat{x} + R\cos(\varphi')\hat{z}\end{aligned} \tag{4}$$

we have

$$\begin{aligned}|\vec{r} - \vec{r}'| &= \sqrt{(x-x')^2 + (y-y')^2 + (y-y')^2}\\ &= \left[(r\sin(\theta)\sin(\varphi) - R\sin(\varphi'))^2 + r^2\cos^2(\theta) + (r\sin(\theta)\cos(\varphi) - R\cos(\varphi'))^2\right]^{1/2}\\ &= \left[r^2\sin^2(\theta)\sin^2(\varphi) + R^2\sin^2(\varphi') - 2rR\sin(\theta)\sin(\varphi)\sin(\varphi') + r^2\cos^2(\theta)\right.\\ &\quad \left.+ r^2\sin^2(\theta)\cos^2(\varphi) + R^2\cos^2(\varphi') - 2rR\sin(\theta)\cos(\varphi)\cos(\varphi')\right]^{1/2}\\ &= \sqrt{r^2 + R^2 - 2rR\sin(\theta)(\cos(\varphi)\cos(\varphi') + \sin(\varphi)\sin(\varphi'))}\\ &= \sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')}\end{aligned} \tag{5}$$

where we used the relation

$$\cos(\varphi \pm \varphi') = \cos(\varphi)\cos(\varphi') \mp \sin(\varphi)\sin(\varphi'). \tag{6}$$

2. Find the charge densities for the two sides of the ring.

Solution: The linear charge density

$$\lambda = \frac{\text{charge}}{\text{length}} \Rightarrow \left[\begin{array}{ll} \lambda_+ = \frac{q}{\pi R}, & \text{for } 0 < \varphi' < \pi \\ \lambda_- = -\frac{q}{\pi R}, & \text{for } \pi < \varphi' < 2\pi \end{array} \right. \tag{7}$$

3. Show that the potential at a point in space described by the vector $\vec{r}$ can be expressed as

$$V(\vec{r}) = \frac{q}{4\pi^2 \epsilon_0 \sqrt{r^2 + R^2}} \left[\int_0^{\pi} \frac{d\varphi'}{\sqrt{1 - u\cos(\varphi - \varphi')}} - \int_0^{\pi} \frac{d\phi}{\sqrt{1 + u\cos(\varphi - \phi)}} \right], \tag{8}$$

where

$$u = \frac{2rR\sin(\theta)}{r^2 + R^2}. \tag{9}$$

Solution: The electric potential for a linear charge density is determined by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{Length} \frac{\lambda dr'}{|\vec{r} - \vec{r}'|} \tag{10}$$

where $dr\prime$ is an infinitessimal length over on the ring which is given by

$$dr' = \sqrt{dx' + dy'} = \sqrt{R^2 \sin^2(\varphi')(d\varphi')^2 + R^2\cos^2(\varphi')(d\varphi')^2} = Rd\varphi'. \tag{11}$$

Using results in parts (a), (b), and the expression for dr', one can express the potential as

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda R d\varphi'}{\sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')}} \\ &= \frac{1}{4\pi\epsilon_0} \int_0^{\pi} \frac{\frac{q}{\pi R} R d\varphi'}{\sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')}} + \int_{\pi}^{2\pi} \frac{\frac{q}{\pi R} R d\varphi'}{\sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')}} \\ &= \frac{q}{4\pi^2\epsilon_0} \left\{ \int_0^{\pi} \frac{d\varphi'}{\sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')}} - \int_{\pi}^{2\pi} \frac{d\varphi'}{\sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')}} \right\} \end{aligned} \tag{12}$$

For the second integral in the above expression we introduce the transformation defined by

$$\begin{aligned} \phi' = \varphi' - \pi \Rightarrow \cos(\varphi - \varphi') = \cos\left(\varphi - \phi' - \pi\right) = -\cos\left(\varphi - \phi'\right) \\ d\varphi' = d\phi', \text{ and } \varphi' = \pi \Rightarrow \phi' = 0, \varphi' = 2\pi \Rightarrow \phi' = \pi, \end{aligned} \tag{13}$$

so that we may write the potential as

$$V(\vec{r}) = \frac{q}{4\pi^2\epsilon_0} \left\{ \int_0^{\pi} \frac{d\varphi'}{\sqrt{r^2 + R^2 - 2rR\sin(\theta)\cos(\varphi - \varphi')}} - \int_0^{\pi} \frac{d\phi'}{\sqrt{r^2 + R^2 + 2rR\sin(\theta)\cos\left(\varphi - \phi'\right)}} \right\} \tag{14}$$

which can be put in the form

$$V(\vec{r}) = \frac{q}{4\pi^2\epsilon_0\sqrt{r^2 + R^2}} \left\{ \int_0^{\pi} \frac{d\varphi'}{\sqrt{1 - \frac{2rR}{r^2+R^2}\sin(\theta)\cos(\varphi - \varphi')}} - \int_0^{\pi} \frac{d\varphi'}{\sqrt{1 + \frac{2rR}{r^2+R^2}\sin(\theta)\cos(\varphi - \varphi')}} \right\}, \tag{15}$$

where we replaced the dummy variable ϕ' by another dummy variable φ'. Introducing the variable

$$u = \frac{2rR\sin(\theta)}{r^2 + R^2} \tag{16}$$

the potential becomes

$$V(\vec{r}) = \frac{q}{4\pi^2 \epsilon_0 \sqrt{r^2 + R^2}} \left[\int_0^{\pi} \frac{d\varphi'}{\sqrt{1 - u\cos(\varphi - \varphi')}} - \int_0^{\pi} \frac{d\phi}{\sqrt{1 + u\cos(\varphi - \phi)}} \right]. \tag{17}$$

4. Using series expansion show that

(a) the potential in (3) can be put in the form

$$V(\vec{r}) = \frac{q}{4\pi^2\epsilon_0\sqrt{r^2+R^2}} \sum_{n=0}^{\infty} \begin{pmatrix} -1/2 \\ n \end{pmatrix} u^n \int_0^{\pi} \left(\left(-1\right)^n - 1\right) \cos^n\left(\varphi - \varphi'\right) d\varphi', \tag{18}$$

where

$$\begin{pmatrix} -1/2 \\ n \end{pmatrix} = \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)...\left(-\frac{1}{2}-n+1\right)}{n!} \tag{19}$$

(b) Show that the potential in (a) near the axis where $\sin(\theta) << 1$ the potential can be approximated as

$$V(\vec{r}) \simeq -\frac{qR}{\pi^2\epsilon_0} \frac{r\sin(\theta)\sin(\varphi)}{\left(r^2+R^2\right)^{3/2}}. \tag{20}$$

(c) Express the result in (b) using Cartesian coordinates

(d) Find the approximate relation for the potential near the center of the ring.

(e) Find the electric field near the axis of the ring.

Solution:

(a) Applying the series expansion

$$(1+x)^p = \sum_{n=0}^{\infty} \begin{pmatrix} p \\ n \end{pmatrix} x^n = 1 + px + \frac{p(p-1)}{2!}x^2 + \text{...convergent for all } |x| < 1 \tag{21}$$

we have

$$\begin{aligned} \left([1 - u\cos(\varphi - \varphi')\right]^{-1/2} &= \sum_{n=0}^{\infty} \begin{pmatrix} -1/2 \\ n \end{pmatrix} (-1)^n u^n \cos^n(\varphi - \varphi') \\ &= 1 - \frac{1}{2}\left(-u\cos(\varphi - \varphi')\right) + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!}u^2\cos^2(\varphi - \varphi') + ... \\ \Rightarrow \left([1 - u\cos(\varphi - \varphi')\right]^{-1/2} &= 1 + \frac{1}{2}u\cos(\varphi - \varphi') + \frac{5}{8}u^2\cos^2(\varphi - \varphi') + ... \end{aligned} \tag{22}$$

and

$$\begin{aligned} \left([1 + u\cos(\varphi - \varphi')\right]^{-1/2} &= \sum_{n=0}^{\infty} \begin{pmatrix} -1/2 \\ n \end{pmatrix} u^n \cos^n(\varphi - \varphi') \\ &= 1 - \frac{1}{2}\left(u\cos(\varphi - \varphi')\right) + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!}u^2\cos^2(\varphi - \varphi') + ... \\ \Rightarrow \left([1 - u\cos(\varphi - \varphi')\right]^{-1/2} &= 1 - \frac{1}{2}u\cos(\varphi - \varphi') + \frac{5}{8}u^2\cos^2(\varphi - \varphi') + ... \end{aligned} \tag{23}$$

so that the potential can be expressed as

$$V(\vec{r}) = \frac{q}{4\pi^2\epsilon_0\sqrt{r^2+R^2}} \sum_{n=0}^{\infty} \begin{pmatrix} -1/2 \\ n \end{pmatrix} u^n \int_0^{\pi} \left[(-1)^n - 1\right] \cos^n(\varphi - \varphi') d\varphi' \tag{24}$$

Noting that

$$(-1)^n - 1 = \left[\begin{array}{ll} 0 & \text{for } n = 0, 2, 4.. \\ -2, & \text{for } n = 1, 3, 5.. \end{array}\right. \tag{25}$$

and expressing the odd numbers as $2m+1$, where $m = 0, 1, 2...$, one can rewrite the potential as

$$V(\vec{r}) = -\frac{q}{2\pi^2\epsilon_0\sqrt{r^2+R^2}} \sum_{m=0}^{\infty} \begin{pmatrix} -1/2 \\ 2m+1 \end{pmatrix} u^{2m+1} \int_0^{\pi} \cos^{2m+1}(\varphi - \varphi') d\varphi' \tag{26}$$

(b) We note that Near the axis where $\sin(\theta) << 1$, since $u << 1$, we can keep only the none zero first order term in the series which is for $m = 0$,

$$V(\vec{r}) \simeq -\frac{q}{2\pi^2\epsilon_0\sqrt{r^2+R^2}}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} u \int_0^{\pi} \cos(\varphi-\varphi')\, d\varphi' = \frac{q}{2\pi^2\epsilon_0\sqrt{r^2+R^2}}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} u \sin(\varphi-\varphi')\Big|_0^{\pi}$$

$$\Rightarrow V(\vec{r}) \simeq \frac{q}{2\pi^2\epsilon_0\sqrt{r^2+R^2}}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} u\left[\sin(\varphi-\pi) - \sin(\varphi)\right] \tag{27}$$

Noting that

$$\sin(\varphi-\pi) - \sin(\varphi) = -2\sin(\varphi), \tag{28}$$

and according to Eq. (19)

$$\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = -\frac{1}{2}, \tag{29}$$

one finds for the potential

$$V(\vec{r}) \simeq \frac{qu\sin(\varphi)}{2\pi^2\epsilon_0\sqrt{r^2+R^2}} = \frac{qR}{\pi^2\epsilon_0}\frac{r\sin(\theta)\sin(\varphi)}{(r^2+R^2)^{3/2}} \tag{30}$$

where we replaced

$$u = \frac{2rR\sin(\theta)}{r^2+R^2} \tag{31}$$

(c) Recalling that

$$x = r\sin(\theta)\sin(\varphi), y = r\cos(\theta), z = r\sin(\theta)\cos(\varphi) \tag{32}$$

and near the axis $\sin(\theta) << 1$, we have

$$r = \sqrt{x^2+y^2+z^2} \simeq y, r\sin(\theta)\sin(\varphi) = x$$

so that

$$V(\vec{r}) \simeq \frac{qR}{\pi^2\epsilon_0}\frac{x}{(y^2+R^2)^{3/2}}. \tag{33}$$

(d) Near the center of the ring where $r << R$ and $\sin(\theta) \simeq 1$, we have

$$u \simeq \frac{2rR}{r^2+R^2} \simeq \frac{2r}{R} \tag{34}$$

and still $u << 1$, our approximation for the potential is valid. But this time since

$$\sin(\theta) \simeq 1, \cos(\theta) \simeq 0$$

we have

$$x = r\sin(\varphi), y = 0, z = r\cos(\varphi) \tag{35}$$

and near the axis $\sin(\theta) << 1$, we have

$$r = \sqrt{x^2+y^2+z^2} \simeq \sqrt{x^2+z^2},$$

and the potential is given by

$$V(\vec{r}) \simeq \frac{qR}{\pi^2\epsilon_0}\frac{x}{(x^2+z^2+R^2)^{3/2}} \tag{36}$$

(e) Using the potential

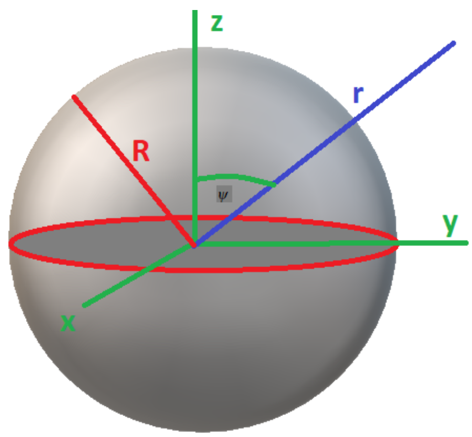
$$V(\vec{r}) \simeq \frac{qR}{\pi^2\epsilon_0}\frac{x}{(y^2+R^2)^{3/2}}. \tag{37}$$

the electric field components in Cartesian coordinates are

$$
\begin{aligned}
E_x(\vec{r}) &= -\frac{\partial}{\partial x}\left[\frac{qR}{\pi^2\epsilon_0}\frac{x}{\left(y^2+R^2\right)^{3/2}}\right] = -\frac{p}{\pi^2\epsilon_0\left(y^2+R^2\right)^{3/2}}, \\
E_y(\vec{r}) &= -\frac{\partial}{\partial y}\left[\frac{qR}{\pi^2\epsilon_0}\frac{x}{\left(y^2+R^2\right)^{3/2}}\right] = \frac{3p}{\pi^2\epsilon_0}\frac{xy}{\left(y^2+R^2\right)^{5/2}}, \\
E_z(\vec{r}) &= -\frac{\partial}{\partial z}\left[\frac{qR}{\pi^2\epsilon_0}\frac{x}{\left(y^2+R^2\right)^{3/2}}\right] = 0.
\end{aligned} \tag{38}
$$

5. *Magnetostatic*

(a) A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω about the z-axis. Find the vector potential and the magnetic field both inside and outside the sphere.



Hint: Rotate the position $\vec{r}$ where we want to determine the vector potential by an angle ψ in a counter clockwise direction so that it coincides with the positive $z-$axis as shown in the figure below.

(b) Applying the result in part (a), find the magnetic field of a uniformly magnetized sphere of radius R and magnetization, $\vec{M} = M\hat{z}$.

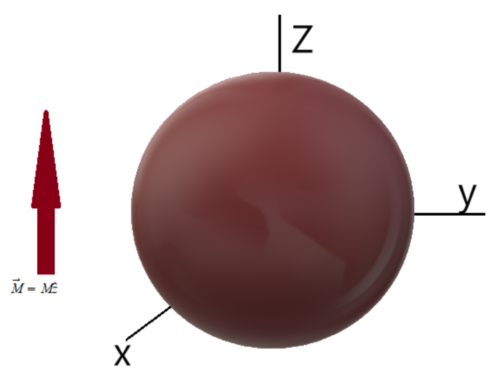


Figure 2: Uniformly magnetized sphere.

Hint: Discuss what a uniform magnetization along the z direction mean in terms of the bound current density.

Solution:

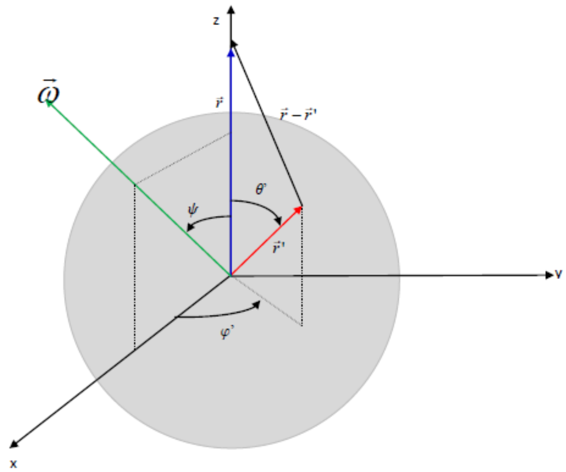


Figure 3: A rotated coordinate.

(a) Let's rotate the position $\vec{r}$ where we want to determine the vector potential by an angle ψ in a counter clockwise direction so that it coincides with the positive $z-$axis as shown in the figure below.

The angular frequency $\vec{\omega}$ lies on the $x-z$ plane and can be expressed as

$$\vec{\omega} = \omega \sin\psi \hat{x} + \omega \cos\psi \hat{z}.$$

The velocity of a charge q located at $\vec{r}' = R\sin\theta'\cos\varphi'\ \hat{x} + R\sin\theta'\sin\varphi'\hat{y} + R\cos\theta'\hat{z}$ on the surface of the sphere can then be expressed as

$$\begin{aligned} \vec{v} = \vec{\omega}\times\vec{r}' &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\varphi' & R\sin\theta'\sin\varphi' & R\cos\theta' \end{vmatrix} \\ &= -R\omega\cos\psi\sin\theta'\sin\varphi'\hat{x} + R\omega\left(\cos\psi\sin\theta'\cos\varphi' - \sin\psi\cos\theta'\right)\hat{y} + R\omega\sin\psi\sin\theta'\sin\varphi'\hat{z}. \end{aligned} \tag{39}$$

The surface current density will then be

$$\begin{aligned} \vec{K}' = \sigma\vec{v} = &-\sigma R\omega\cos\psi\sin\theta'\sin\varphi'\hat{x} + \sigma R\omega\left(\cos\psi\sin\theta'\cos\varphi' - \sin\psi\cos\theta'\right)\hat{y} \\ &+\sigma R\omega\sin\psi\sin\theta'\sin\varphi'\hat{z}. \end{aligned} \tag{40}$$

Noting that

$$|\vec{r}-\vec{r}'| = \sqrt{R^2+r^2-2rR\cos\theta'},\, da' = R^2\sin\theta' d\theta' d\varphi' \tag{41}$$

the vector potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{sur}\frac{\vec{K}(r')}{|\vec{r}-\vec{r}'|}da' \tag{42}$$

becomes

$$\begin{aligned} \vec{A} = \frac{\mu_0\sigma}{4\pi}\Bigg\{ &-\int_0^{\pi}\int_0^{2\pi}\frac{R^3\omega\cos\psi\sin^2\theta'\sin\varphi' d\theta' d\varphi'}{\sqrt{R^2+r^2-2rR\cos\theta'}}\hat{x} + \int_0^{\pi}\int_0^{2\pi}\frac{R^3\omega\left(\cos\psi\sin^2\theta'\cos\varphi' - \sin\psi\sin\theta'\cos\theta'\right)}{\sqrt{R^2+r^2-2rR\cos\theta'}}d\theta' d\varphi'\hat{y} \\ &+\int_0^{\pi}\int_0^{2\pi}\frac{R^3\omega\sin\psi\sin^2\theta'\sin\varphi' d\theta' d\varphi'}{\sqrt{R^2+r^2-2rR\cos\theta'}}\hat{z}\Bigg\}. \end{aligned} \tag{43}$$

Because of the integrals

$$\int_0^{2\pi}\sin\varphi' d\varphi' = \int_0^{2\pi}\cos\varphi' d\varphi' = 0 \tag{44}$$

the terms involving $\sin\varphi'$ and $\cos\varphi'$ vanish when we integrate over φ'. Hence

$$\vec{A} = -\frac{\mu_0\sigma}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{R^3\omega\sin\psi\sin\theta'\cos\theta' d\theta' d\varphi'}{\sqrt{R^2+r^2-2rR\cos\theta'}}\hat{y} = -\frac{\mu_0\sigma R^3\omega\sin\psi}{2}\int_0^{\pi}\frac{\sin\theta'\cos\theta' d\theta'}{\sqrt{R^2+r^2-2rR\cos\theta'}}\hat{y}. \tag{45}$$

Introducing the transformation of variable defined by

$$u = \sqrt{R^2+r^2-2rR\cos\theta'} \Rightarrow \begin{cases} \cos\theta' = \frac{R^2+r^2-u^2}{2rR}, \\ \frac{\sin\theta'\cos\theta' d\theta'}{\sqrt{R^2+r^2-2rR\cos\theta'}} = \frac{\left(R^2+r^2-u^2\right)du}{2(rR)^2}, \\ \theta = 0 \Rightarrow u = \sqrt{R^2+r^2-2rR} = |r-R|\,, \\ \theta = \pi \Rightarrow u = \sqrt{R^2+r^2+2rR} = r+R, \end{cases} \tag{46}$$

we find

$$\vec{A} = -\frac{\mu_0\sigma R^3\omega\sin\psi}{2}\int_{|r-R|}^{r+R}\frac{\left(R^2+r^2-u^2\right)du}{2\left(rR\right)^2}\hat{y} = -\frac{\mu_0\sigma R\omega\sin\psi}{4r^2}\left[\left(R^2+r^2\right)u-\frac{u^3}{3}\right]_{|r-R|}^{r+R}\hat{y}$$

$$\Rightarrow \vec{A} = -\frac{\mu_0\sigma R\omega\sin\psi}{4r^2}\left[\left(R^2+r^2\right)u-\frac{u^3}{3}\right]_{|r-R|}^{r+R}\hat{y} \tag{47}$$

We need to consider two cases. The first is when we are outside the sphere (i.e. $r > R \Rightarrow |r-R| = r-R$), which gives

$$\vec{A} = -\frac{\mu_0\sigma R\omega\sin\psi}{4r^2}\left\{\left[R^2+r^2-\frac{(r+R)^2}{3}\right](r+R)-\left[R^2+r^2-\frac{(r-R)^2}{3}\right](r-R)\right\}\hat{y} \tag{48}$$

which can be simplified into

$$\vec{A}\left(\vec{r}\right) = -\frac{\mu_0\sigma R^4\omega\sin\psi}{3r^2}\hat{y}. \tag{49}$$

The second case is when we are inside the sphere (i.e. $r < R \Rightarrow |r-R| = -\left(r-R\right)$, the vector potential becomes

$$\vec{A} = -\frac{\mu_0\sigma R\omega\sin\psi}{4r^2}\left\{\left[R^2+r^2-\frac{(r+R)^2}{3}\right](r+R)+\left[R^2+r^2-\frac{(r-R)^2}{3}\right](r-R)\right\}\hat{y} \tag{50}$$

which also can be simplified to give

$$\vec{A}\left(\vec{r}\right) = -\frac{\mu_0\sigma Rr\omega\sin\psi}{3}\hat{y}. \tag{51}$$

Therefore the vector potential is given by

$$\vec{A} = \begin{cases} -\frac{\mu_0\sigma Rr\omega\sin\psi}{3}\hat{y} & r<R \\ -\frac{\mu_0\sigma R^4\omega\sin\psi}{3r^2}\hat{y} & r>R \end{cases} = \begin{cases} \frac{\mu_0\sigma R}{3}\vec{\omega}\times\vec{r} & r<R \\ \frac{\mu_0\sigma R^4}{3r^3}\vec{\omega}\times\vec{r} & r>R \end{cases}. \tag{52}$$

where we used

$$\vec{\omega}\times\vec{r} = -r\omega\sin\psi\hat{y} \tag{53}$$

referring to Fig. 3

(b) To find the magnetic field we first need to find the vector potential due to the bound currents. Since the material has a uniform magnetization the volume current density is zero

$$\vec{J}_b\left(\vec{r}\right) = \nabla\times\vec{M}\left(\vec{r}\right) = 0.$$

The magnetization $\vec{M}$ pointing along the z direction, in spherical coordinates (Fig. 4, can be expressed as

$$\vec{M} = M\hat{z} = M\cos\left(\theta\right)\hat{r} - M\sin\left(\theta\right)\hat{\theta}$$

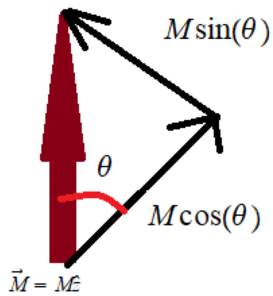


Figure 4: The components of the magnetization in spherical coordinates.

and the normal unit vector to the area is

$$\hat{n} = \hat{r}. \tag{54}$$

Then the surface current

$$\vec{K}_b(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n} = \left(M\cos(\theta)\,\hat{r} - M\sin(\theta)\,\hat{\theta}\right) \times \hat{r}$$
$$\Rightarrow \vec{K}_b(\vec{r}) = M\sin(\theta)\,\hat{\varphi}. \tag{55}$$

We recall from Example 5.11 the vector potential for a spherical shell of radius, R, with surface charge density, σ, and spinning about the z axis with angular velocity ω, generates a surface current given by

$$\vec{K} = \sigma\vec{v} = \sigma R\omega\sin(\theta)\,\hat{\varphi} \tag{56}$$

which lead us to a vector potential given by

$$\vec{A} = \begin{cases} \frac{\mu_0\sigma R}{3}\omega r\sin(\theta)\,\hat{\varphi} & r < R \\ \frac{\mu_0\sigma R^4}{3r^2}\omega\sin(\theta)\,\hat{\varphi} & r > R \end{cases}. \tag{57}$$

Comparing Eq. (55) with (56), we have $\sigma R\omega = M$ and the vector potential in Eq. (57) becomes

$$\vec{A} = \begin{cases} \frac{\mu_0 M}{3} r\sin(\theta)\,\hat{\varphi} & r < R \\ \frac{\mu_0 M R^3}{3r^2}\sin(\theta)\,\hat{\varphi} & r > R \end{cases}. \tag{58}$$

Then using the expression for the magnetic field in terms of the vector potential in spherical coordinates

$$\vec{B}(\vec{r}) = \nabla \times A = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial}{\partial\varphi}(A_\theta)\right]\hat{r}$$
$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(A_r) - \frac{\partial}{\partial r}(rA_\varphi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right]\hat{\varphi}, \tag{59}$$

inside the sphere ($r < R$), we find

$$\vec{B}(\vec{r}) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\mu_0 M}{3}r\sin(\theta)\right)\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(r\frac{\mu_0 M}{3}r\sin(\theta)\right)\right]\hat{\theta}$$
$$= \frac{2\mu_0 M}{3}\left[\cos(\theta)\,\hat{r} - \sin(\theta)\,\hat{\theta}\right]. \tag{60}$$

Similarly, outside the sphere ($r > R$)

$$\vec{B}(\vec{r}) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\mu_0 M R^3}{3r^2}\sin(\theta)\right)\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(r\frac{\mu_0 M R^3}{3r^2}\sin(\theta)\right)\right]\hat{\theta}$$
$$\Rightarrow \vec{B}(\vec{r}) = \frac{2\mu_0 M R^3}{3r^3}\cos(\theta)\,\hat{r} + \frac{\mu_0 M R^3}{3r^3}\sin(\theta)\,\hat{\theta}. \tag{61}$$

In terms of the Magnetization vector

$$\vec{M} = M\hat{z} = M\cos(\theta)\,\hat{r} - M\sin(\theta)\,\hat{\theta} \Rightarrow M\sin(\theta)\,\hat{\theta} = M\cos(\theta)\,\hat{r} - \vec{M}, \tag{62}$$

one can rewrite the magnetic field, inside the sphere, as

$$\vec{B}(\vec{r}) = \frac{2\mu_0\vec{M}}{3} \tag{63}$$

and outside the sphere

$$\begin{aligned}\vec{B}(\vec{r}) &= \frac{2\mu_0 MR^3}{3r^3}\cos(\theta)\,\hat{r} + \frac{\mu_0 R^3}{3r^3}\left(M\cos(\theta)\,\hat{r} - \vec{M}\right) \\ &= \frac{3\mu_0 MR^3\cos(\theta)\,\hat{r} - \mu_0 R^3\vec{M}}{3r^3} \Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 R^3\left(\left(3\vec{M}\cdot\hat{r}\right)\hat{r} - \vec{M}\right)}{3r^3}.\end{aligned} \tag{64}$$

Noting that for a sphere of radius R with a uniform magnetization $\vec{M}$, in terms of the total magnetic dipole moment, $\vec{m}_{total}$,

$$M = \frac{m_{total}}{\frac{4}{3}\pi R^3} \Rightarrow \vec{M} = \frac{\vec{m}_{total}}{\frac{4}{3}\pi R^3}. \tag{65}$$

one can write the magnetic field outside the sphere as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left[\left(3\vec{m}_{total}\cdot\hat{r}\right)\hat{r} - \vec{m}_{total}\right]. \tag{66}$$