PHYS 4330 ELECTRICITY & MAGNETISM II

HOMEWORK ASSIGNMENT 01

DUE DATE: February 11, 2020

Instructor: Dr. Daniel Erenso Name: ________________________

Mandatory problems: *Any two of the problems*

Student signature:____

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1. A copper rod of length, h, mass, m, and electric resistance, R, slides with negligible friction on metal rails that have negligible electric resistance (see Fig. 1). The rails are connected on the right with a wire of negligible electric resistance, and a magnetic compass is placed under this wire. There is a uniform magnetic field, $\vec{B}$, pointing out of the page that fills the entire region.

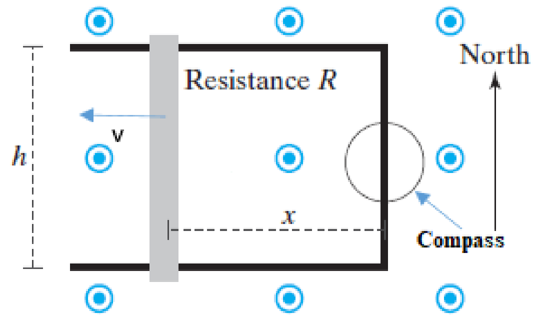


Figure 1: A rod sliding on u-shaped wire in a uniform magnetic field.

(a) If the rod moves to the left at a speed v, what is the magnitude and direction of the current in the circuit? Which direction would the compass deflect?

(b) What is the magnetic force on the bar?

(c) If the rod starts out with speed, v_0, at time, $t = 0$, and is left to slide, what is its speed at a later time, t.

(d) The initial kinetic energy of the bar was, of course,

$$KE_i = \frac{1}{2} m v_0. \tag{1}$$

Show that the energy delivered to the resistor is exactly equal to this initial kinetic energy.

Solution:

(a) As the rod moves to the left, the flux is increasing and therefore there must be an induced magnetic field in the opposite direction of the external magnetic field to oppose the increase in the flux. This requires an induced current flow in a clockwise direction. This current flows down along the right side of the wire. This results in an induced magnetic field directed from left to right under this wire. and therefore the compass would deflect away from the wire. In order to find the magnitude of the current we need to find the magnitude of the induced voltage

$$I_{in} = \frac{V_{in}}{R}. \tag{2}$$

From faraday's law the induced voltage is given by

$$V_{in} = \frac{d\Phi}{dt} \tag{3}$$

The flux through the area bounded by h and $x(t)$ is given by

$$\Phi(t) = x(t) h B \tag{4}$$

Noting that $x(t) = vt$, we have

$$\Phi(t) = Bhvt \tag{5}$$

So that

$$V_{in} = \frac{d}{dt}(hvt) = hvB \tag{6}$$

and the magnitude of the current becomes

$$I_{in} = \frac{hvB}{R}. \tag{7}$$

(b) The magnetic force on the rod is given by

$$F_m = I_{in} h B = \frac{h^2 v B^2}{R}. \tag{8}$$

Since the induced current direction is upward in the rod, using right hand rule, one finds the direction of the force to be to the right.

(c) Using Newtons second law

$$m\frac{dv}{dt} = -F_m \tag{9}$$

one can find the speed of the rod as a function of time,

$$m\frac{dv}{dt} = -\frac{h^2 v B^2}{R} \Rightarrow \frac{dv}{v} = -\frac{h^2 B^2}{mR}dt \Rightarrow \int\limits_{v_0}^{v(t)} \frac{dv}{v} = -\frac{h^2 B^2}{mR}t$$

$$\Rightarrow v(t) = v_0 e^{-\frac{h^2 B^2 t}{mR}} \tag{10}$$

Clearly the rod is slowing down by the magnetic force and eventually comes to stop.

(d) The energy delivered to the resistor per unit time (power) can be determined from

$$P = \frac{dW}{dt} = {I_{in}}^2 (t) R. \tag{11}$$

Using Eqs. (7) and (10), we can write

$$\frac{dW}{dt} = \frac{h^2 v^2 (t) B^2}{R^2} R = \frac{B^2 h^2 v_0^2}{R} e^{-\frac{2h^2 B^2 t}{mR}}. \tag{12}$$

Then the total energy delivered to the resistor

$$W = \frac{B^2 h^2 v_0^2}{R} \int\limits_0^\infty e^{-\frac{2h^2 B^2 t}{mR}} dt \Rightarrow W = \frac{B^2 h^2 v_0^2}{R} \cdot \frac{mR}{2h^2 B^2}$$

$$\Rightarrow W = \frac{1}{2} m v_o^2 \tag{13}$$

which is exactly the same as the kinetic energy lost by the rod.

2. A conducting circular loop of radius, b, is placed in a uniform magnetic field pointing along the z-direction, $\vec{B} = B_0 \hat{z}$ and rotates with an angular velocity, ω, about a diameter which is perpendicular to $\vec{B}_0$ (i.e. $\vec{\omega} = -\omega \hat{x}$) (See Fig.2). Find the current in the loop, the retarding torque, and the average power which is required to maintain the rotation.

Solution: At instant of time the angle between the normal to the area bounded by the ring is $\varphi = \omega t$. The component of the $\vec{B}$ field normal to the area is

$$B_\perp = B_0 \cos(\omega t). \tag{14}$$

Then the using the flux

$$\Phi(t) = \pi a^2 B_\perp = \pi a^2 B_0 \cos(\omega t) \tag{15}$$

the magnitudes of the induced induced voltage can be written as

$$V_{in} = \left| \frac{d\Phi}{dt} \right| = \pi \omega a^2 B_0 \sin(\omega t). \tag{16}$$

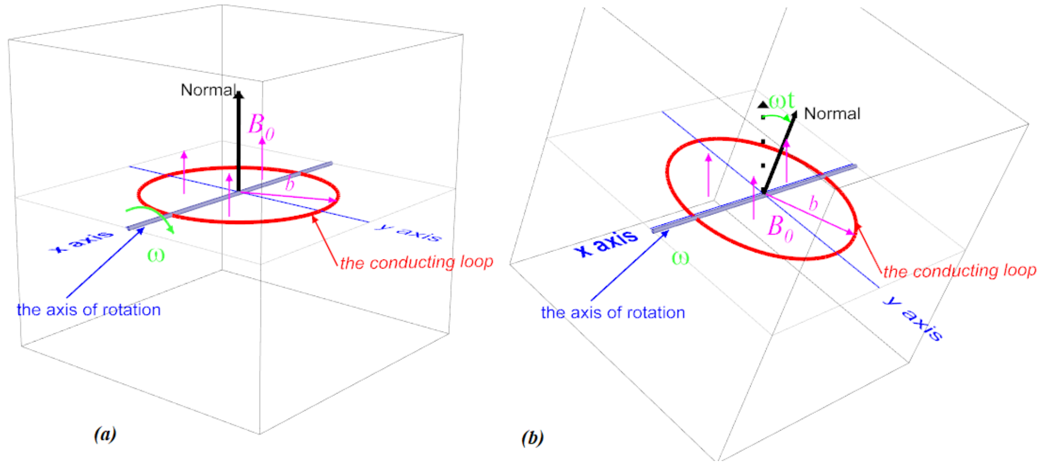


Figure 2: (a) The ring on the x-y plane at $t = 0$. (b) after it rotated about the x-axis at time, t. The angular displacement at this time is ωt from the z-axis.

Assuming the resistance of the wire is R, the induced current becomes

$$I_{in} = \frac{V_{in}}{R} = \frac{\pi \omega a^2 B_0}{R} \sin(\omega t). \tag{17}$$

The retarding torque is due to the magnetic moment created by the induced current. This can be determined from

$$\vec{\tau} = \vec{m} \times \vec{B}. \tag{18}$$

At a given instant of time the magnetic moment can be expressed as

$$\vec{m} = \pi a^2 I_{in} \hat{n} \tag{19}$$

Noting that from Fig. 2 the unit normal vector to the area at a given instant of time, can be expressed as

$$\hat{n} = \sin(\omega t)\,\hat{y} + \cos(\omega t)\,\hat{z} \tag{20}$$

one can write the torque as

$$\vec{\tau} = \pi a^2 I_{in} \left(\sin(\omega t)\,\hat{y} + \cos(\omega t)\,\hat{z}\right) \times B_0 \hat{z} = \pi a^2 B_0 I_{in} \sin(\omega t)\,\hat{x} \tag{21}$$

and substituting the induced current

$$\vec{\tau} = \left(\frac{\pi a^2 B_0}{R}\right)^2 \omega \sin^2(\omega t)\,\hat{x} \tag{22}$$

The average power required to maintain the rotation is

$$\langle P \rangle = \left\langle \frac{dW}{dt} \right\rangle = \langle V_{in}(t)\, I_{in}(t) \rangle. \tag{23}$$

Using the results we determined

$$\langle P \rangle = \frac{\left(\pi \omega a^2 B_0\right)^2}{R} \left\langle \sin^2(\omega t) \right\rangle. \tag{24}$$

and noting that

$$\left\langle \sin^2(\omega t) \right\rangle = \frac{1}{T} \int_0^T \sin^2(\omega t)\, dt = \frac{1}{T} \int_0^{2\pi/T} \sin^2\left(\frac{2\pi}{T} t\right) dt = \frac{1}{2} \tag{25}$$

one can easily show that

$$\langle P \rangle = \frac{\left(\pi \omega a^2 B_0\right)^2}{2R}. \tag{26}$$

3. A square loop is cut out of a thick sheet of aluminium. It is then placed so that the top portion is in a uniform magnetic field, $\vec{B}$, and is allowed to fall under gravity. The shaded region in the diagram (Fig.3) shows the magnetic field region which points out of the page (i.e. $\vec{B} = -B_0\hat{y}$). If the magnetic field is $1T$,

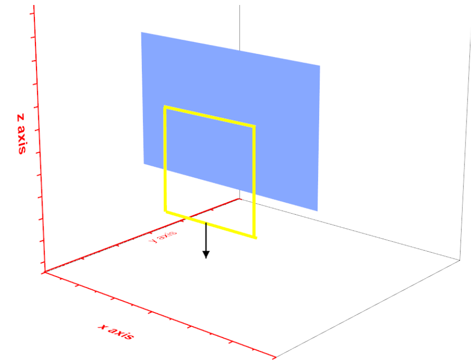


Figure 3: A square conducting loop falling in a uniform magnetic field.

(a) find the terminal velocity of the loop (in m/s).

(b) find the velocity of the loop as a function of time.

(c) how long does it take (in seconds) to reach, say 90% of the terminal speed? What would happen if you cut a tiny slit in the square loop, breaking the circuit?

Solution:

(a) As the square loop is falling down the flux decreases and there will be an induced voltage that leads to an induced current. The induced current must oppose the decrease in flux. Since the magnetic field is directed out of the page, in order to oppose the decrease in magnetic flux, the induced magnetic field must be directed out of the page and therefore The induced current must flow in a counterclockwise direction. So it must flow in clockwise direction. Due to this current, the top side of the conducting loop experiences a net magnetic force directed upward. This magnetic force is given by

$$F_m = IaB, \tag{27}$$

where

$$I = \frac{V_{in}}{R} = \frac{1}{R}\left|\frac{d\phi}{dt}\right|. \tag{28}$$

Noting that the flux is given by

$$\phi(t) = a\,(a - y(t))\,B. \tag{29}$$

where a is the length of the sides of the square loop and y is length of the square outside the magnetic field region. For a free-fall, we have

$$\frac{dv}{dt} = g \Rightarrow v = gt \Rightarrow y\,(t) = \frac{1}{2}gt^2, \tag{30}$$

so that

$$\phi(t) = a\left(a - \frac{1}{2}gt^2\right)B. \tag{31}$$

Then the induced voltage (magnitude)

$$V_{in} = \left|\frac{d\phi}{dt}\right| = agBt \tag{32}$$

in terms of the velocity we may write the time, t, as

$$v = gt \Rightarrow t = \frac{v}{g} \tag{33}$$

and the induced voltage can be expressed as

$$V_{in} = agB\frac{v}{g} = avB \Rightarrow I = \frac{avB}{R}. \tag{34}$$

Then for the magnetic force one can write

$$F_m = IaB = \frac{a^2vB^2}{R}. \tag{35}$$

and the terminal velocity

$$F_m = F_{gravity} \Rightarrow \frac{a^2v_tB^2}{R} = mg \Rightarrow v_t = \frac{mgR}{a^2B^2}. \tag{36}$$

(b) At any given instant of time, the equation of motion for the square loop is given by Newton's second law

$$F_{net} = F_g - F_m = m\frac{dv}{dt}$$

$$\Rightarrow mg - \frac{a^2vB^2}{R} = m\frac{dv}{dt} \Rightarrow 1 - \frac{a^2vB^2}{mgR} = \frac{1}{g}\frac{dv}{dt} \Rightarrow \int\limits_0^{v(t)} \frac{dv}{1-\alpha v} = g\int\limits_0^t dt. \tag{37}$$

$$\int\limits_0^{v(t)} \frac{dv}{1-\alpha v} = g\int\limits_0^t dt \qquad \text{(Eq. 1)}$$

where we introduced the constant

$$\alpha = \frac{a^2B^2}{mgR}.$$

Integrating Eq. (1), we find.

$$-\frac{\ln\left(1-\alpha v\left(t\right)\right)}{\alpha} = gt \Rightarrow e^{-\alpha gt} = 1 - \alpha v\left(t\right) \Rightarrow v\left(t\right) = \frac{1}{\alpha}\left(1 - e^{-\alpha gt}\right)$$

or

$$v(t) = \frac{mgR}{a^2B^2}\left(1 - e^{-\frac{a^2B^2}{mgR}t}\right)$$

In terms of the terminal velocity

$$v_t = \frac{mgR}{a^2B^2}$$

and the constant

$$\beta = \frac{a^2B^2}{mR}$$

we may write

$$v\left(t\right) = v_t\left(1 - e^{-\beta t}\right).$$

The time at which $v(t) = 0.9v_t$ is given by

$$0.9v_t = v_t\left(1 - e^{-\beta t}\right) \Rightarrow e^{-\beta t} = 0.1 \Rightarrow t = -\frac{\ln(0.1)}{\beta}$$

To get the numerical values for v_t and t, use

$$m = 4\eta Aa$$

where η is the mass density of aluminium

$$\eta = 2.7\times 10^3\ \frac{kg}{m^3}$$

and A is the cross-sectional area. The resistance of the square loop can expressed as

$$R = \frac{4a\rho}{A}$$

where ρ is the resistivity of aluminium

$$\rho = 2.8 \times 10^{-8} dm$$

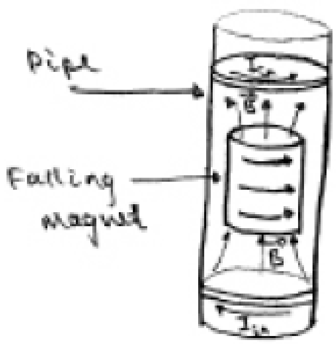
and we will find

$$v_f = 1.2 cm/s, t = 2.8 ms.$$

(c) If the loop were cut it would fall freely with acceleration g.

4. Griffiths Problem 7.14 (*remember the demo I showed you in class*)

Solution: Suppose the current (I) in the magnet flows counterclockwise (viewed from above), as shown, so it's field, near the ends, points upward. A ring of pipe below the magnet experiences an increasing upward flux as the magnet approaches, and hence (by Lenz's law) a current (I_{ind}) will be induced in it such as to produce a downward flux.



Thus (I_{ind}) must flow clockwise, which is opposite to the current in the magnet. Since opposite currents repel, the force on the magnet is upward. Meanwhile, a ring above the magnet experiences a decreasing (upward) flux, so its induced current is parallel to I, and it attracts the magnet upward. And the flux through rings next to the magnet is constant, so no current is induced in them. Conclusion: the delay is due to forces exerted on the magnet by induced eddy currents in the pipe.

5. An alternating current $I = I_{\max}\cos(\omega t)$ flows down a long straight wire with negligible diameter and returns along a coaxial conducting tube of radius b as shown in Fig. 4

(a) In what direction does the induced electric field point. Give your answer using unit vectors in cylindrical coordinates (radial ($\pm\hat{s}$), circumferential ($\pm\hat{\varphi}$), or axial ($\pm\hat{z}$).

(b) Assuming that the field goes to zero as $s \to \infty$, find $\vec{E}(s,t)$.

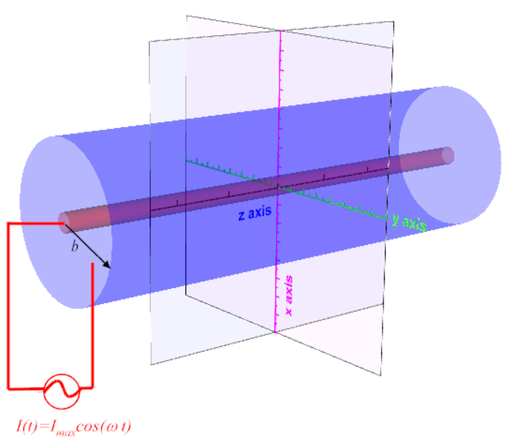
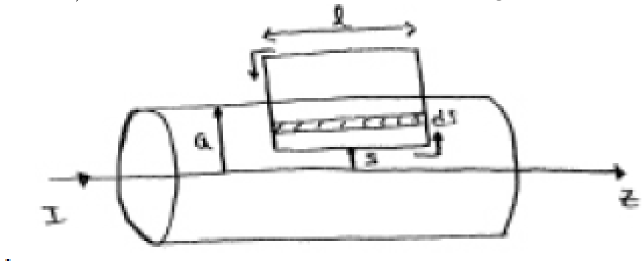


Figure 4: An ac current flows down a long wire (red) and returns along a coaxial conducting tube (blue).

Solution:

(a) The magnetic field (in the quasistatic approximation) is "circumferential". This is analogous to the current in



a solenoid, and hence the field is longitudinal.

(b) Use the "Amperian loop" shown in the figure above. Outside $\vec{B}$ is zero, so here $\vec{E} = 0$ (like $\vec{B}$ outside a solenoid). This leads to

$$\begin{aligned}
\oint \vec{E} d\vec{l} &= El = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \vec{B} d\vec{a} \\
El &= -\frac{d}{dt}\int_s^a \frac{\mu_0 I}{2\pi s'} l ds' \\
E &= -\frac{d}{dt}\left[\frac{\mu_0 I}{2\pi s}\ln\left(\frac{a}{s}\right)\right] \\
E &= -\frac{d}{dt}\left[\frac{\mu_0 I\cos(\omega t)}{2\pi s}\ln\left(\frac{a}{s}\right)\right] \\
E &= \frac{\mu_0 I_0 \omega}{2\pi}\ln\left(\frac{a}{s}\right)\sin(\omega t)\,\hat{z}
\end{aligned}$$

6. *Problem 1 with a twist*: A copper bar of length h and electric resistance R slides with negligible friction on metal rails that have negligible electric resistance (see Fig.5). The rails are connected on the right with a wire of negligible electric resistance, and a magnetic compass is placed under this wire. The compass needle deflects to the right of north, as shown on the diagram. Throughout this region there is a uniform magnetic field $\vec{B}$ pointing out of the page, produced by large coils that are not shown. This magnetic field is increasing with time, and the magnitude is, $B = B_0 + \alpha t$, where B_0 and α are constants, and t is the time in seconds. You slide the copper bar to the right and at time, $t = 0$, you release the bar when it is a distance x from the right end of the apparatus. At that instant the bar is moving to the right with a speed v.

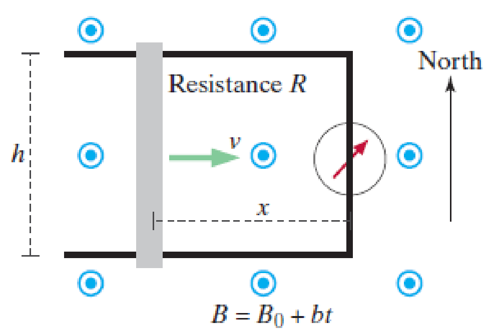


Figure 5: A rod sliding on u-shaped wire in a time varying magnetic field. The magntic field is increasing at a rate, $\frac{dB}{dt} = \alpha$

(a) Calculate the magnitude of the initial current I in this circuit.

(b) Calculate the magnitude of the net force on the bar just after you release it.

Solution:

(a)

(b)

7. *Similar to Example 7.9*: A line charge is glued onto the rim of a wheel of radius, b, which is then suspended horizontally, as shown in the figure below, so that it is free to rotate (the spokes are made of some nonconducting material) (see Fig. 6). In the central region, out to radius a, there is a magnetic field pointing along the positive z-axis ($\vec{B} = B_0\hat{z}$). This magnetic field begins to increase at a rate of

$$\frac{dB}{dt} = \alpha.$$

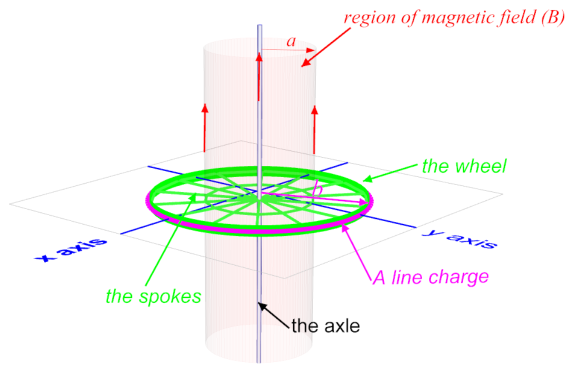


Figure 6: A circular (radius b) line charge (pink) glued to none conducting wheel (on the x-y plane) connected by a none conducting spokes (green) is free to rotate about an axle. A uniform magnetic field along the z-direction confined to a region with radius, a (red) is increasing at a rate, $\frac{dB}{dt} = \alpha$, where α is a constant.

(a) Explain qualitatively and quantitatively what is going to happen?

(b) Show that the angular momentum imparted to the wheel does not depend on the value of α.

Solution:

(a)

(b)

8. In Fig . 7 there is a solenoidal coil with radius, b. A thin conducting ring of radius, a, (i.e. $a < b$) is sitting under the solenoid. The solenoid is connected to a switched-off power supply. You suddenly turned the switch on.

(a) If the ring has a resistance, R, what would be the magnitude and direction of the induced current in the ring? Both the axial and radial components of the magnetic field of the solenoid can be approximated to be a constant.

(b) Find the approximate magnitude and the direction of the magnetic force on the ring.

Solution:

(a)

(b)

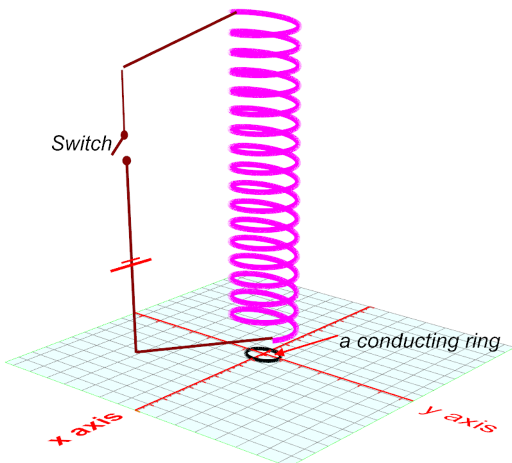


Figure 7: A solenoidal, a ring sitting at the bottom of the solenoid. The two ends of the solenoid are connected to a power suply with a switch.