PHYS 4330 ELECTRICITY & MAGNETISM II

HOMEWORK ASSIGNMENT 02

DUE DATE: February 18, 2020

Instructor: Dr. Daniel Erenso Name: ____________________

Mandatory problems: *Any two of the problems*

Student signature:____

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P #	1	2	3	4	5	Score
Score	/	/	/	/	/	/100

1. A small loop of radius, b is held a distance z_0 above the center of a larger loop with radius a. The small loop carries a current, I_1, in a clockwise direction and the larger loop carries a current, I_2, in a counterclockwise direction (both viewed from the top). The plane of the two loops are parallel and also perpendicular to the z-axis. (See Fig.1)

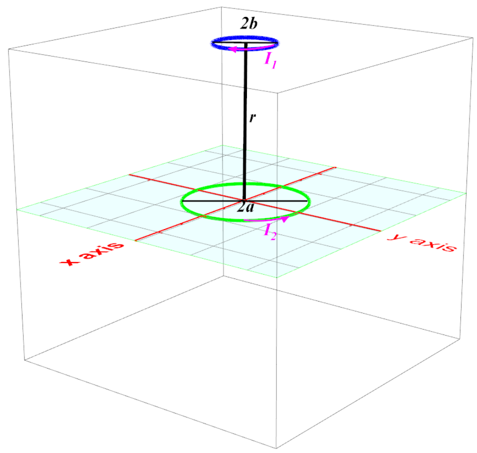


Figure 1: Two circular current carrying wires.

(a) Find the flux through the little loop. (The little loop is so small and you may consider the field of the big loop to be essentially constant.)

(b) Find the flux through the larger loop. (The little loop is so small that you may treat it as a magnetic dipole.)

(c) Find the mutual inductance and confirm that $M_{12} = M_{21}$.

Solution:

(a) If a current I_2 flows in the big loop, the magnetic field due to this current at a distance r from the center of the loop is given by

$$\vec{B}(r) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + r^2)^{(3/2)}} \hat{z}, \tag{1}$$

where a is the radius of the circular loop. Then the flux through the little loop is

$$\Phi = \int \vec{B} \cdot d\vec{a} = \left(\frac{\mu_0 I}{2} \frac{a^2}{(a^2 + r^2)^{(3/2)}} \right) \pi b^2 \Rightarrow \Phi = \frac{\pi \mu_0 I a^2 b^2}{2\,(b^2 + z^2)^{(3/2)}}. \tag{2}$$

here the magnetic field is taken to be constant over the area of the little loop.

(b) The little loop is so small that it can be taken as a magnetic dipole and its magnetic field is expressible as

$$\vec{B}(r) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left(3\,(\vec{m} \cdot \hat{r})\,\hat{r} - \vec{m} \right),$$

where (referring to the figure above)

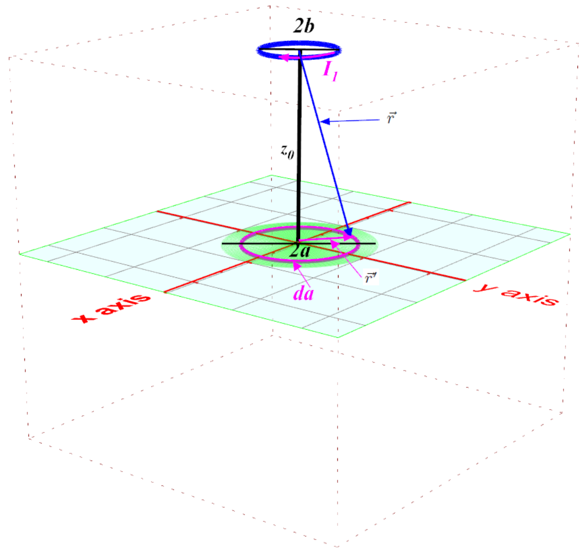
$$r = \sqrt{z_o^2 + r'^2} \Rightarrow \hat{r} = \frac{\vec{r}}{r} = \frac{\vec{r}' - z_o \hat{z}}{r}$$

and

$$d\vec{a} = 2\pi r' dr' \left(-\hat{z} \right).$$

Then flux becomes

$$\Phi = \int \vec{B} \cdot d\vec{a} = \frac{\mu_0}{4\pi} \int \frac{1}{(z_o^2 + r'^2)^{(3/2)}} \left[3\,(\vec{m} \cdot \hat{r})\,\hat{r} - \vec{m} \right] \cdot d\vec{a}.$$



Noting that

$$\vec{m} = m\,(-\hat{z}) = I_1 \pi b^2\,(-\hat{z})\,, \tag{3}$$

we have

$$\vec{m}\cdot\hat{r} = I_1\pi b^2\,(-\hat{z})\cdot\left[\frac{\vec{r'} - z_o\hat{z}}{r}\right]$$

and noting that in cylindrical coordinates

$$\vec{r'} = r'\hat{s} \Rightarrow r'\hat{z}\cdot\hat{s} = 0,$$

we find

$$\begin{aligned}
\vec{m}\cdot\hat{r} &= \frac{I_1\pi b^2 z_o}{r} = \frac{I_1\pi b^2 z_o}{\sqrt{z_o^2 + r'^2}},\\
\hat{r}\cdot d\vec{a} &= \left(\frac{r'\hat{s} - z_o\hat{z}}{\sqrt{z_o^2 + r'^2}}\right)\cdot(2\pi r'dr')\,(-\hat{z}) = \frac{2\pi z_o r' dr'}{\sqrt{z_o^2 + r'^2}},\\
\vec{m}\cdot d\vec{a} &= I_1\pi b^2(-\hat{z})\cdot(2\pi r'dr')\,(-\hat{z}) = 2I_1\pi^2 b^2 r' dr'.
\end{aligned}$$

The magnetic flux can then be expressed as

$$\begin{aligned}
\Phi &= \frac{\mu_0}{4\pi}\int \frac{1}{(z_o^2 + r'^2)^{(3/2)}}\left[3\,(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}\right]\cdot d\vec{a}\\
&= \frac{\mu_0}{4\pi}\int \frac{1}{(z_o^2 + r'^2)^{(3/2)}}\left[3\frac{I_1\pi b^2 z_o}{\sqrt{z_o^2 + r'^2}}\cdot\frac{2\pi z_o r' dr'}{\sqrt{z_o^2 + r'^2}} - 2I_1\pi^2 b^2 r' dr'\right]\\
&= \frac{\mu_0}{4\pi}\left[6I_1\pi^2 b^2 z_o^2\int_0^a \frac{r'dr'}{(z_o^2 + r'^2)^{(5/2)}} - 2I_1\pi^2 b^2\int_0^a \frac{r'dr'}{(z_o^2 + r'^2)^{(3/2)}}\right]\\
&= \frac{\mu_0}{4\pi}\left[6I_1\pi^2 b^2 z_o^2\left.\frac{\left(z_o^2 + r'^2\right)^{\frac{-3}{2}}}{-3}\right|_0^a - 2I_1\pi^2 b^2\left.\frac{\left(z_o^2 + r'^2\right)^{\frac{-1}{2}}}{-2}\right|_0^a\right] = \frac{\mu_0}{4\pi}2I_1\pi^2 b^2\left[-z_o^2\left(z_o^2 + a^2\right)^{\frac{-3}{2}} + \left(z_o^2 + a^2\right)^{\frac{-1}{2}}\right]\\
&= \frac{\pi\mu_0 I_1 b^2}{2}\left[\frac{1}{(z_o^2 + a^2)^{\frac{1}{2}}} - \frac{z_o^2}{(z_o^2 + a^2)^{\frac{3}{2}}}\right] \Rightarrow \Phi = \frac{\pi\mu_0 I_1 a^2 b^2}{2\,(z_o^2 + a^2)^{\frac{3}{2}}}.
\end{aligned}$$

Which is exactly the same as the flux through the little loop as one should expect.

(c) Noting that the flux through the little loop can be expressed as

$$\Phi = MI$$

we find that

$$M_{12} = M_{21} = M = \frac{\pi \mu_0 a^2 b^2}{2\left(z^2 + b^2\right)^{\frac{3}{2}}}$$

2. A long straight conductor carrying a current, I_1, and ring of radius, a, carrying a current, I_2, lie in the same plane as shown in Fig. 2 (i.e. y-z plane). The distance between the wire and the center of the ring is b. Find the mutual inductance M and force F between the two conductors.

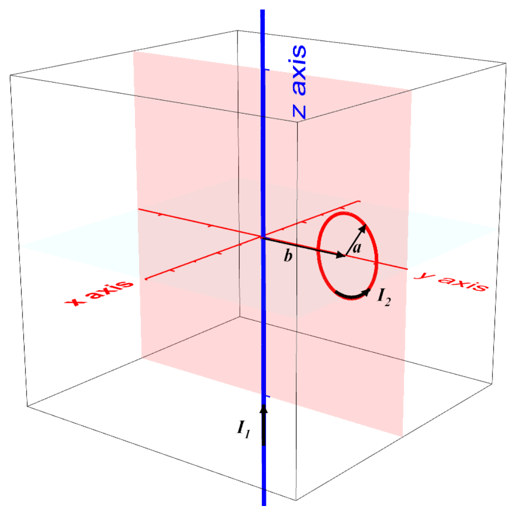
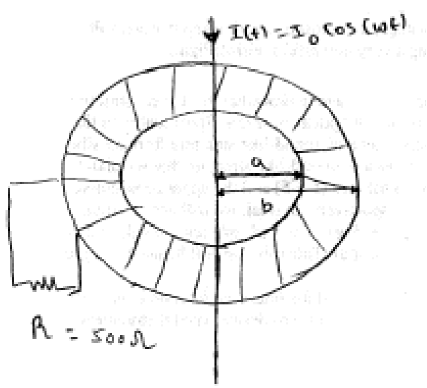


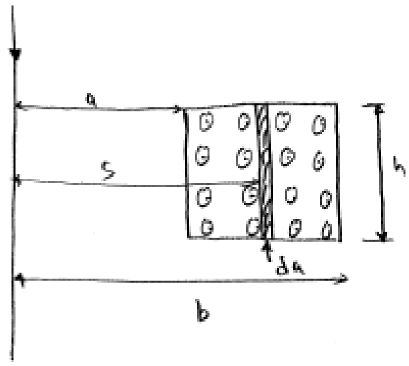
Figure 2: A long wire and a circular loop on the y-z plane.

3. Griffiths Problem 7.26



Sol:

(a) If we cut the toroid right at the center and look at the cross section, (just at one of the rectangular loops) it looks like the following:



The flux through a single rectangular loop due to the current in the straight wire is given by

$$\Phi_1 = \int \vec{B} \cdot d\vec{a}$$

and using

$$B = \frac{\mu_0 I}{2\pi s}, da = hds$$

we find

$$\Phi_1 = \frac{\mu_0 I h}{2\pi} \int_a^b \frac{ds}{s} \Rightarrow \Phi_1 = \frac{\mu_0 I h}{2\pi} \ln\left(\frac{b}{a}\right).$$

Then the total flux through the N turns becomes

$$\Phi_T = N\Phi_1 = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right).$$

The induced EMF

$$\varepsilon_{in} = -\frac{d\Phi_T}{dt} \Rightarrow \varepsilon_{in} = -\frac{\mu_0 N h}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI}{dt}$$

Using $I = I_0 \cos(\omega t)$, we find

$$\varepsilon_{in} = \frac{\mu_0 N I_0 \omega h}{2\pi} \ln\left(\frac{b}{a}\right) \sin(\omega t) \Rightarrow \varepsilon_{in} = 2.61 \times 10^{-4} \sin(\omega t).$$

Then the current

$$I_r = \frac{\varepsilon_{in}}{R} = \frac{\mu_0 N I_0 \omega h}{2\pi R} \ln\left(\frac{b}{a}\right) \sin(\omega t) \Rightarrow I = 5.22 \times 10^{-7} \sin(\omega t). \qquad \text{(Eq. 1)}$$

(b) If a current I_r flows in a toroid, then the magnetic field due to this current will be

$$B = \begin{cases} \frac{\mu_0 N I_r}{2\pi S} & \text{for } a < s < b \\ 0 & \text{outside} \end{cases}$$

This field produces a "self" flux given by

$$\Phi_S = N \int \vec{B} \cdot d\vec{a} = \frac{\mu_0 N^2 I_r}{2\pi} \int_a^b \frac{hds}{s} \Rightarrow \Phi_S = \frac{\mu_0 N^2 I_r h}{2\pi} \ln\left(\frac{b}{a}\right)$$

Then the back EMF will be

$$\varepsilon_{back} = -\frac{d\Phi_S}{dt} = -\frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI_r}{dt}. \qquad \text{(Eq. 2)}$$

Using Eq. (1), we find

$$\frac{dI_r}{dt} = \frac{\mu_0 N I_0 \omega^2 h}{2\pi R} \ln\left(\frac{b}{a}\right) \cos(\omega t)$$

so that the back EMF becomes

$$\varepsilon_{back} = \frac{\mu_0^2 N^3 \omega^2 h^2}{4\pi^2 R}\left[\ln\left(\frac{b}{a}\right)\right]^2 \cos(\omega t) \Rightarrow \varepsilon_{back} = -2.74\times 10^{-7}\cos(\omega t).$$

Then the ratio of the back and direct (induced) EMTs will be

$$\frac{\varepsilon_{back}}{\varepsilon_{in}} = \frac{\frac{\mu_0^2 N^3\omega^2 h^2}{4\pi^2 R}\left[\ln\left(\frac{b}{a}\right)\right]^2}{\frac{\mu_0 N I_0 \omega h}{2\pi}\ln\left(\frac{b}{a}\right)} \Rightarrow \frac{\varepsilon_{back}}{\varepsilon_{in}} = \frac{\mu_0 N^2 \omega h \ln\left(\frac{b}{a}\right)}{2\pi R} = 1.05\times 10^{-3}$$

4. Griffiths Problem 7.29

Sol: The energy stored in a magnetic field is given by

$$W = \frac{1}{2\mu_0}\int B^2 d\tau$$

where the magnetic field of a toroid is given by

$$B = \begin{cases} \frac{\mu_0 N I}{2\pi S} & \text{for } a < s < b \\ 0 & \text{outside} \end{cases}$$

The infinitesimal volume in the region $a < s < b$ can be expressed as

$$d\tau = 2\pi s h ds$$

Then the energy

$$W = \frac{1}{2\mu_0}\int_a^b \frac{(\mu_0)^2 N^2 I^2 h}{(2\pi)^2 s^2} 2\pi s h ds \Rightarrow W = \frac{\mu_0 N^2 I^2 h}{4\pi}\int_a^b \frac{ds}{s}$$

$$\Rightarrow W = \frac{\mu_0 N^2 I^2 h}{4\pi}\ln\left(\frac{b}{a}\right).$$

Using the relation

$$W = \frac{1}{2}LI^2 \Rightarrow L = \frac{2W}{I^2} \Rightarrow L = \frac{\mu_0 N^2 h}{4\pi}\ln\left(\frac{b}{a}\right)$$

which is exactly the same as Eq. (7.27)

5. Griffiths Problem 7.32

Solution:

(a) To find the mutual inductance, first let's find the flux through area $\vec{a}_2$ due to the magnetic field of a current I, in the first loop. The loops are tiny and can be treated as a dipole with a dipole moment.

$$\begin{aligned}\vec{m}_1 &= I_1\vec{a}_1 \\ \vec{m}_2 &= I_2\vec{a}_2\end{aligned}$$

The magnetic field due to $\vec{m}_1$ at the position of the second loop can then be written as

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1\right], \vec{B}_1 = \frac{\mu_0 I_1}{4\pi}\frac{1}{r^3}\left[3(\vec{a}_1\cdot\hat{r})\hat{r} - \vec{a}_1\right].$$

The flux through the second loop

$$\Phi_2 = \int_{a_2}\vec{B}_1\cdot d\vec{a}_2 \simeq \vec{B}_1\cdot\vec{a}_2 \Rightarrow \Phi_2 = \frac{\mu_0 I_1}{4\pi}\frac{1}{r^3}\left[3(\vec{a}_1\cdot\hat{r})(\vec{a}_2\cdot\hat{r}) - \vec{a}_1\cdot\vec{a}_2\right].$$

But we also know

$$\Phi_2 = M_{12} I_2 \Rightarrow M_{12} = \frac{\Phi_2}{I_2}$$

which leads to

$$M_{12} = \frac{\mu_0}{4\pi r^3} \left[3\left(\vec{a}_1 \cdot \hat{r}\right)\left(\vec{a}_2 \cdot \hat{r}\right) - \vec{a}_1 \cdot \vec{a}_2\right].$$

Similarly, the flux through loop 1 due to the current in loop 2, I_2, can be written as

$$\begin{aligned} \Phi_1 &= \vec{B}_2 \cdot \vec{a}_1 = \frac{\mu_0 I_2}{4\pi} \frac{1}{r^3} \left[3\left(\vec{a}_2 \cdot \hat{r}\right)\left(\vec{a}_1 \cdot \hat{r}\right) - \vec{a}_1 \cdot \vec{a}_2\right] \\ \Phi_1 &= M_{21} I_2 \Rightarrow M_{21} = \frac{\mu_0}{4\pi r^3} \left[3\left(\vec{a}_2 \cdot r\right)\left(\vec{a}_1 \cdot r\right) - \vec{a}_1 \cdot \vec{a}_2\right] = M_{12} = M \end{aligned}$$

$$M_{21} = \frac{\mu_0}{4\pi r^3} \left[3\left(\vec{a}_2 \cdot r\right)\left(\vec{a}_1 \cdot r\right) - \vec{a}_1 \cdot \vec{a}_2\right] = M_{12} = M \qquad \text{(Eq. 1)}$$

(b) The work done per unit time against the mutually induced EMF to keep the current I_1 flowing in loop 1 can be expressed as

$$\frac{dW}{dt} = -\varepsilon_{in} I_1 \Rightarrow dW = -\varepsilon_{in} I_1 dt$$

But we know that

$$\varepsilon_{in} = -M \frac{dI_2}{dt}.$$

Then

$$dW = M \frac{dI_2}{dt} I_1 dt \Rightarrow W = M \int_0^{I_2} I_1 dI_2 \Rightarrow W = M I_1 I_2.$$

and the energy, using Eq. (1), can be put in the form

$$W = \frac{\mu_0}{4\pi r^3} \left[3\left(I_2 \vec{a}_2 \cdot \hat{r}\right)\left(I_1 \vec{a}_1 \cdot \hat{r}\right) - \left(I_1 \vec{a}_1\right) \cdot \left(I_2 \vec{a}_2\right)\right]$$

or

$$W = \frac{\mu_0}{4\pi r^3} \left[3\left(\vec{m}_1 \cdot \hat{r}\right)\left(\vec{m}_2 \cdot \hat{r}\right) - \vec{m}_1 \cdot \vec{m}_2\right]$$

Which is the interaction energy of two dipoles.

6. Griffiths Problem 7.33

Sol: