PHYS 4380 Final Exam

December 04, 2018

Instructor: Dr. Daniel Erenso Name:——————————

Instruction: This Final exam has two parts prepared from all the material we covered throughout the semester. Each part worth 50% Part one will be two hours in class exam that will be given on Thursday December 12, 2018 from 1:00pm to 3:00 p.m.. The problems listed in Part I are sample problems for the in-class exam. The two problems listed in Part two are the take-home problems and are due on Thursday December 13, 2018 at 1:00pm. Equations, integrals and constants that you are not expected to memorize will be provided for the in-class part of the exam.

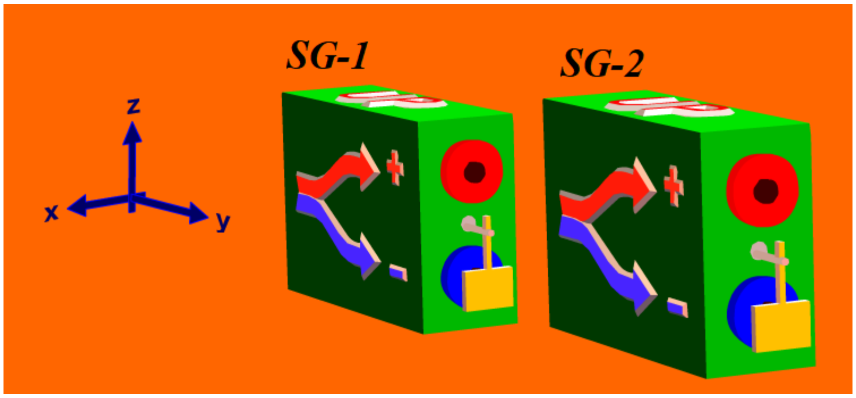
YOU HAVE TWO HOURS TO COMPLETE THIS TEST

	Part I			Part II		
Problem	1	2	3	4	5	Total
Score	/15	/15	/20	/25	/25	/100

Part I **In-class exam practice problems**

1. Provide a short and brief answer

(a) Consider the two SG devices shown in the figure. Both devices have none uniform magnetic field in the z-direction A spin-half particle in a state $|+X\rangle$ is incident on SG-1.



(i) What is the probability of this particle exiting SG-1 in the state $|+Z\rangle$.[2 pts]

(ii) Suppose the particle exiting the SG-1 in a $|+Z\rangle$ state enters the SG-2. What is the probability that this particle exiting SG-2 in a state $|-Z\rangle$. [2 pts]

(iii.) Suppose you rotated SG-2 about the y axis by $\pi/2$, what would be the probability that the particle exiting SG-2 is in a state $|-X\rangle$. [2 pts]

(b) Suppose there are two particles of same type. Particle one is in a state described by the ket vector

$$|\psi_1\rangle = e^{\frac{i\pi}{4}} \left[\frac{1}{2} |a_1\rangle + \frac{\sqrt{3}i}{2} |a_2\rangle \right]$$

and particle two in a state described by the ket vector

$$|\psi_2\rangle = \frac{1}{2} |a_1\rangle + \frac{\sqrt{3}i}{2} |a_2\rangle ,$$

where the vectors $\{|a_1\rangle, |a_2\rangle\}$ form an orthonormal set of basis vectors. Let the operator $\hat{A}$ represent some measurable physical observable such that

$$\hat{A}|a_1\rangle = A_1 |a_1\rangle , \hat{A}|a_2\rangle = A_2 |a_2\rangle .$$

Would the average values and the uncertainties for the measurement of this observable for these two particles be the same or different? Explain why! [5 pts]

(c) Explain briefly the similarities and differences of the two virtual SG experiments we discussed in class and shown in the figures below. [5 pts]

2. The polarization state for a photon propagating in the z-direction is given by

$$|\psi\rangle = e^{i\theta} \left[\sqrt{\frac{2}{3}} |X\rangle + \frac{i}{\sqrt{3}} |Y\rangle \right]$$

(a) What is the probability that a photon in this state will pass through an ideal polarizer with its transmission axis oriented in the $|Y\rangle$ direction? [5 pts]

(b) What is the probability that a photon in this state will pass through an ideal polarizer with its transmission axis oriented in the $|Y'\rangle$ making an angle ϕ with the $|Y\rangle$ axis? [5 pts]

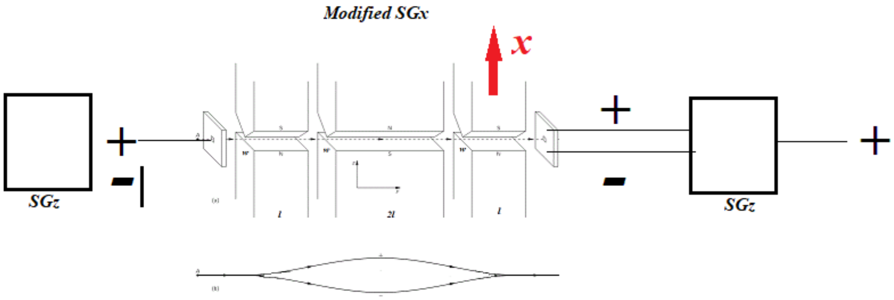


Figure 1: Modified SGx device with both states ($|+X\rangle$ and $|-X\rangle$) open.

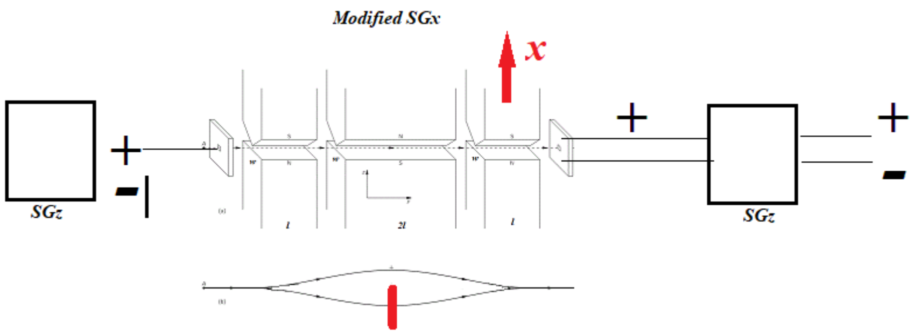


Figure 2: Modified SGx apparatus with $|+X\rangle$ open and $|-X\rangle$ blocked.

(c) Determine the state $|\psi\rangle$ in the $|R\rangle$ and $|L\rangle$ basis. [5 pts]

(d) A beam carrying N photons per second, each in the state $|\psi\rangle$, is totally absorbed by a black disk with its normal to the surface in the z direction. How large is the torque,

$$\vec{\tau} = \frac{d\vec{J}}{dt}$$

exerted on the disk by the photons with

(i) right circular polarization (photons in the state $|R\rangle = \frac{1}{\sqrt{2}}(|X\rangle + i\,|Y\rangle)$) [5 pts]

(ii) left circular polarization (photons in the state $|L\rangle = \frac{1}{\sqrt{2}}(|X\rangle - i\,|Y\rangle)$) [5 pts]

Note: The magnitude of the angular momentum of a photon is $\hbar$.

(e) Suppose the photons described by the state $|\psi\rangle$ are produced from a linearly polarized light (at an angle of 45° to the x and y axes) of wavelength 5890 $\mathring{A}$ incident normally on a birefringent crystal that has its optic axis parallel to the face of the crystal along the x-axis If the index of refraction for light of this wavelength polarized along y (perpendicular to the optic axis) is 1.66 and the index of refraction for the light polarized along x (parallel to the optic axis) is 1.49, what should be the length of the crystal that produces photons in the state $|\psi\rangle$.

3. For a spin half particle described by the state

$$|+n\rangle = \cos\left(\frac{\theta}{2}\right)|+Z\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|-Z\rangle$$

(a) Express the state vector in S_y basis and S_x basis [5 pts]

(b) Find the probability amplitudes that the measurement for S_y yields $+\hbar/2$ and $-\hbar/2$? [5 pts]

(c) What are the probabilities that the measurement for S_y yields $+\hbar/2$ and $-\hbar/2$? [5 pts]

(d) Find the expectation values $\left\langle \hat{S}_y \right\rangle$, $\left\langle \hat{S}_y^2 \right\rangle$, and the uncertainty ΔS_y. [10 pts]

4. The general uncertainty principle states that for two physical observables represented by operators $\hat{A}$ and $\hat{B}$, the uncertainties in simultaneous measurement of these physical observables is determined by the inequality

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left| \left\langle \hat{C} \right\rangle \right|^2,$$

where the operator $\hat{C}$ is related by the commutation relation of the two operators given by

$$\left[\hat{A}, \hat{B}\right] = i\hat{C}.$$

(a) Using the commutation relations for the components of the angular momentum operators

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z, \left[\hat{J}_y, \hat{J}_z\right] = i\hbar\hat{J}_x, \left[\hat{J}_z, \hat{J}_x\right] = i\hbar\hat{J}_y$$

determine the uncertainty relations for simultaneous measurements of any two components of the angular momentum. [10 pts]

(b) Suppose the operator $\hat{A}$ is the x-component for the position operator, $\hat{x}$, and $\hat{B}$ is the x-component for momentum operator, $\hat{p}_x$, given by

$$\hat{p}_x = -i\hbar\frac{d}{dx}, \hat{x} = x$$

by deriving the commutation relation for

$$\left[\hat{A}, \hat{B}\right] = i\hat{C}.$$

find the Heisenberg uncertainty relation for momentum and position. [15 pts]

5. At time $t = 0$, an electron and a positron are formed in a state with total spin angular momentum equal to zero. The particles are situated in a uniform magnetic field $\vec{B}_0 = B_0\hat{z}$

(a) In the absence of interaction between the electron and the positron show that the spin Hamiltonian of the system may be written as

$$\hat{H} = \omega_0 \left(\hat{S}_{1z} - \hat{S}_{2z}\right)$$

where

$$\omega_0 = \frac{eB_0}{mc}$$

[2 pts]

(b) What is the Spin state of the system at a time t [6 points]

(c) At time t, measurements are made for $\hat{S}_{1x}$ and $\hat{S}_{2x}$. Calculate the probability that both of these measurements yield the value $-\hbar/2$ [7 pts].

(d) In the presence of interaction between the electron and the positron the spin Hamiltonian may be written as

$$\hat{H} = \frac{2A}{\hbar}\vec{S}_1 \cdot \vec{S}_2 + \omega_0 \left(\hat{S}_{1z} - \hat{S}_{2z}\right).$$

Determine the energy eigenvalues [10 pts]

6. A quantum harmonic oscillator under the influence of an external electric field can be described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + eE\cos(\omega_0 t)\,\hat{x}$$

(a) Determine the Heisenberg equation of motion for the operators $\hat{p}_x(t)$ and $\hat{x}(t)$. Note: both $\hat{p}_x(t)$ and $\hat{x}(t)$ do not explicitly depend on time,

$$\frac{\partial \hat{p}_x(t)}{\partial t} = \frac{\partial \hat{x}(t)}{\partial t} = 0.$$

[5 pts]

(b) Show that the equation of motion is just the classical equation of motion for a harmonic oscillator subjected to a periodic force. Solve for $\hat{p}(t)$ and $\hat{x}(t)$ in terms of $\hat{p}(0)$ and $\hat{x}(0)$.[10 pts]

(c) Express the quantum Hamiltonian in terms of the Ladder operators

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}} \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}} \end{aligned} \tag{1}$$

[5 pts]

(d) Find the equations of motion for the Ladder operators $\hat{a}$ and $\hat{a}^\dagger$ using the result in part (c). [5 pts]

7. Townsend 3.1, 3.8,Townsend 4.4, Townsend 4.14

8. *For* $j = 3/2$

(a) *Determine all the basis states (i.e. the angular momentum eigenstates).*

(b) *Determine the matrix representation of* $\hat{J}_z, \hat{J}_+$, *and* $\hat{J}_-$ *using the corresponding basis states.*

(c) *Use the result* $\hat{J}_+$, *and* $\hat{J}_-$ *in part (b) and find the matrix representation for* $\hat{J}_x$, *and* $\hat{J}_y$

(d) *Solve the eigenvalue equation for* $\hat{J}_y$.

9. Consider an electron (charge, $q = -e$ and mass, m) with magnetic dipole moment $\vec{m}$. Somebody turned on a uniform magnetic field B_0 directed along the positive y-axis.

(a) Determine the classical Hamiltonian

(b) Determine the quantum Hamiltonian

(c) Solve the eigenvalue equation for the quantum Hamiltonian

(d) Determine the state of the electron at a later time t if the initially state of the electron, $|\psi(0)\rangle = |-Z\rangle$ and $|\psi(0)\rangle = |-X\rangle$.

(e) Using the state vector determined in (d) find the probabilities $P_{+z}(t)$ and $P_{-z}(t)$, $P_{+x}(t)$ and $P_{-x}(t)$ for both initial states.

(f) Find the expectation values for $\left\langle \hat{S}_z \right\rangle$, $\left\langle \hat{S}_x \right\rangle$, and $\left\langle \hat{S}_y \right\rangle$ when the electron is initially in the state $|\psi(0)\rangle = |-X\rangle$.

(g) Verify the results you determined in (f) using the Heisenberg equation.

10. A particle in a one dimensional infinite square well potential has the initial wave function

$$\Psi(x,0) = \frac{1}{6}\left[4u_1(z) + 3u_3(z) - u_6(z) + \sqrt{10}u_7(z)\right].$$

(a) What is the expectation value of the energy and position at the initial time? [7 points]

(b) Find the wave function at a later time t, $\Psi(x,t)$ [9 pts]

(c) Determine the expectation values for $\langle \hat{x} \rangle$ and $\left\langle \hat{H} \right\rangle$ at a later time t [8 pts]

11. For a particle of mass in the 1-D potential energy well

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{elsewhere} \end{cases}$$

is at time $t = 0$ in the state

$$\psi(x, t=0) = \begin{cases} \left(\frac{1+i}{2}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right) + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right) & 0 < x < a \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the wave function at a later time, $\psi(x, t)$.

(b) What is the expectation value for the energy, $\left\langle \hat{H} \right\rangle$?

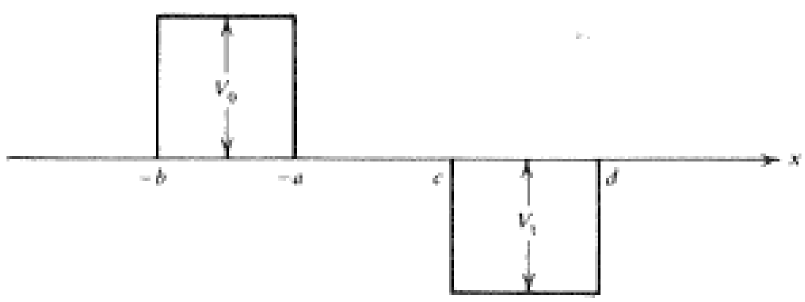
(c) What is the probability that a measurement of the energy will yield the value

$$E = \frac{\pi^2 \hbar^2}{2ma^2}$$

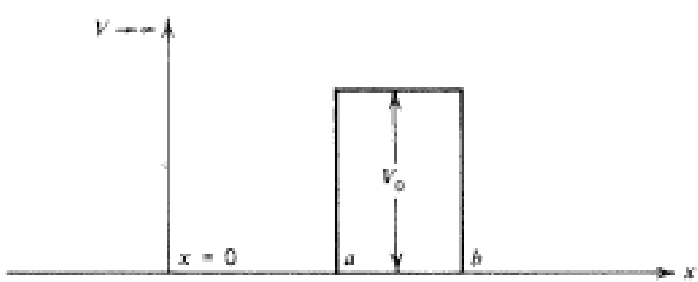
(d) Without detailed computation, give an argument that $\langle x \rangle$ is time dependent.

12. Without actually solving the Schrödinger equation, set-up the solutions so that only the matching of eigen functions and their derivatives remain to be done for the following situations.

(a) Particle is incident from the left and $E < V_0$



(b) Particle is incident from the right and $E < V_0$



Part II: Take-home Problems

4. In nano-physics you will often encounter physicists talk about *quantum dots, quantum wires, and quantum wells.* A simplified model describing a quantum-well is basically an electron that is free to move in any of the two directions in space but confined in the third direction by an infinite potential. Suppose the confinement is in the z-direction, the potential energy is given by

$$V(z) = \begin{cases} 0, & 0 < z < a \\ \infty, & \text{elsewhere} \end{cases}$$

and the electron move freely along the x and y directions. Suppose at $t = 0$ the electron in this quantum well is described by the wave function

$$\Psi(z, 0) = A \sin^3(\pi z/a)..$$

   (a) Determine A and express the initial wave function as a superposition of the energy eigen functions for a particle in one dimensional infinite square well potential [4 pts]

   (b) What is the average energy at the initial time? [4 points]

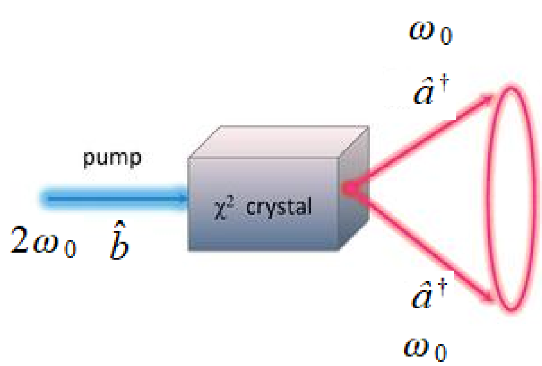
   (c) Find the wave function at a later time, $\psi(z,t)$.[7 points]

   (d) Will the expectation values for the energy $\left\langle \hat{H} \right\rangle$ be different at a later time? Explain why? [3 points]

   (e) Will the expectation values for the position $\langle \hat{z} \rangle$ be different at a later time? Explain why? [4 points]

   (f) Find the expectation value $\langle \hat{p}^2 \rangle$ [4 points]

5. In quantum theory of radiation, a single mode electromagnetic radiation is just a collection of quantum harmonic oscillators. Spontaneous parametric down conversion (SPDC) is a quantum processes that can be described using the quantum theory of the electromagnetic field. In SPDC a pump photon of frequency $2\omega_0$ interacts with a nonlinear medium and is down converted into identical twin photons of frequency, ω_0 This quantum



optical interaction, in the interaction picture, is described by the quantum Hamiltonian

$$\hat{H}_I = \hbar\kappa \left(\hat{a}^{\dagger 2}\hat{b} + \hat{a}^2\hat{b}^\dagger \right),$$

where $\hat{a}^\dagger, \hat{a}$ and $\hat{b}^\dagger, \hat{b}$ are the ladder operators that you were introduced to. But here these operators are referred as photon creation and annihilation operators for the down converted photons ($\hat{a}^\dagger$ and $\hat{a}$, respectively) and for the pump photon ($\hat{b}^\dagger$ and $\hat{b}$, respectively) These operators obey the commutation relations

$$\left[\hat{b}, \hat{b}^\dagger\right] = 1, \left[\hat{a}, \hat{a}^\dagger\right] = 1, \left[\hat{a}, \hat{b}, \right] = 0$$

The constant κ is know as the coupling constant which describes the strength of interaction of the pump photon with the none linear medium. It usually depends on the second order electrical susceptibility (χ^2) of the medium.

(a) Show that the interaction Hamiltonian

$$\hat{H}_I = \hbar\kappa\left(\hat{a}^{\dagger 2}\hat{b} + \hat{a}^2\hat{b}^{\dagger}\right)$$

is Hermitian.[2 points]

(b) In "the parametric approximation", the pump photons are treated classically and the pump depletion is negligible. Under this approximation the operators describing the pump photons can be replaced by a complex variable

$$\hat{b} \rightarrow |\beta|\, e^{-i\varphi}$$

and the interaction Hamiltonian can be written as

$$\hat{H}_I = \hbar\kappa\,|\beta|\left(\hat{a}^{\dagger 2}e^{-i\varphi} + \hat{a}^2 e^{i\varphi}\right).$$

Using this approximate quantum Hamiltonian find the Heisenberg equations of motion for the operators describing the down converted photons (i.e. you determine the equations

$$\frac{d\hat{a}}{dt} = -i\Omega_p\hat{a}^{\dagger}e^{-i\varphi},\ \frac{d\hat{a}^{\dagger}}{dt} = i\Omega_p\hat{a}e^{i\varphi}$$

where

$$\Omega_p = 2\kappa\,|\beta|$$

known as the effective Rabi frequency.) [7 pts]

(c) Show that solution to the Heisenberg equations are

$$\begin{aligned}\hat{a}\,(t) &= \hat{a}\,(0)\cosh\left(\Omega_p t\right) - i\hat{a}^{\dagger}\,(0)\sinh\left(\Omega_p t\right)e^{-i\varphi}\\ \hat{a}^{\dagger}\,(t) &= \hat{a}^{\dagger}\,(0)\cosh\left(\Omega_p t\right) + i\hat{a}\,(0)\sinh\left(\Omega_p t\right)e^{i\varphi}\end{aligned}$$

[7 points]

(d) Suppose initially the down converted photons are in a vacuum state (i.e. no photon at $t = 0$) and the initial state can be expressed as

$$|\psi\,(0)\rangle = |n\rangle\,,$$

where $n = 0$. We recall the Hermitian quadrature operators we were introduced in chapter 4

$$\hat{a}_1 = \frac{\hat{O}^{\dagger} + \hat{O}}{2},\ \hat{a}_2 = \frac{i\left(\hat{O}^{\dagger} - \hat{O}\right)}{2},$$

For the down converted photons the field quadratures can be expressed as

$$\hat{a}_1 = \frac{\hat{a}^{\dagger} + \hat{a}}{2},\ \hat{a}_2 = \frac{i\left(\hat{a}^{\dagger} - \hat{a}^{\dagger}\right)}{2}.$$

Determine the expectation values $\langle\hat{a}_1\rangle\,,\langle\hat{a}_2\rangle\,,\left\langle\hat{a}_1^2\right\rangle\,,\left\langle\hat{a}_2^2\right\rangle$ and show that the uncertainties in the field quadratures, for $\varphi = 0$, are given by

$$\Delta a_1 = \frac{1}{2}e^{-\Omega_p t},\ \Delta a_2 = \frac{1}{2}e^{\Omega_p t}.$$

[9 points]