Instruction: The Midterm exam has two parts. Each part worth 50\% Part one will be 85 minutes in class exam that will be given on Thursday October 11, 2018. The problems listed in Part I are sample problems for the in-class exam. The two problems listed in Part two are the take-home problems and are due October 18. However, it would be for your own good if you turn your solutions before Friday October 12, 2018. Equations, integrals and constants that you are not expected to memorize will be provided for the in-class part of the exam.

YOU HAVE 85 MINUTES TO COMPLETE THIS TEST

|  | Part I |  |  |  | Part II |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Problem | 1 | 2 | 3 | 4 | 5 | Total |  |
| Score | $/ 20$ | $/ 10$ | $/ 20$ | $/ 20$ | $/ 20$ | $/ 100$ |  |

## Part I In-class exam practice problems

1. Provide a short and brief answer
(a) Consider the vector in a complex Cartesian vector space

$$
\vec{A}=3 \hat{x}-4 i \hat{y}
$$

Suppose the unit vectors $\hat{x}, \hat{y}$, and $\hat{z}$ can be represented by $\left|e_{1}\right\rangle,\left|e_{2}\right\rangle$, and $\left|e_{3}\right\rangle$. Express $\vec{A}$ and $\vec{A}^{*}$ using Dirac notation [2 pts]
(b) Consider the two SG devices shown in the figure. Both devices have none uniform magnetic field in the z-direction A spin-half particle in a state $|+X\rangle$ is incident on SG-1.

(i) What is the probability of this particle exiting SG-1 in the state $|+Z\rangle .[2 \mathrm{pts}]$
(ii) Suppose the particle exiting the $\mathrm{SG}-1$ in a $|+Z\rangle$ state enters the $\mathrm{SG}-2$. What is the probability that this particle exiting SG-2 in a state $|-Z\rangle$. [2 pts]
(iii.) Suppose you rotated SG- 2 about the y axis by $\pi / 2$, what would be the probability that the particle exiting SG-2 is in a state $|-X\rangle$. [2 pts]
(c) Suppose there are two particles of same type. Particle one is in a state described by the ket vector

$$
\left|\psi_{1}\right\rangle=e^{\frac{i \pi}{4}}\left[\frac{1}{2}\left|a_{1}\right\rangle+\frac{\sqrt{3} i}{2}\left|a_{2}\right\rangle\right]
$$

and particle two in a state described by the ket vector

$$
\left|\psi_{2}\right\rangle=\frac{1}{2}\left|a_{1}\right\rangle+\frac{\sqrt{3} i}{2}\left|a_{2}\right\rangle,
$$

where the vectors $\left\{\left|a_{1}\right\rangle,\left|a_{2}\right\rangle\right\}$ form an orthonormal set of basis vectors. Let the operator $\hat{A}$ represent some measurable physical observable such that

$$
\hat{A}\left|a_{1}\right\rangle=A_{1}\left|a_{1}\right\rangle, \hat{A}\left|a_{2}\right\rangle=A_{2}\left|a_{2}\right\rangle .
$$

Would the average values and the uncertainties for the measurement of this observable for these two particles be the same or different? Explain why! [5 pts]
(d) Explain briefly the similarities and differences of the two virtual SG experiments we discussed in class and shown in the figures below. [5 pts]
2. The polarization state for a photon propagating in the z-direction is given by

$$
|\psi\rangle=e^{i \theta}\left[\sqrt{\frac{2}{3}}|X\rangle+\frac{i}{\sqrt{3}}|Y\rangle\right]
$$



Figure 1: Modified SGx device with both states $(|+X\rangle$ and $|-X\rangle)$ open.


Figure 2: Modified SGx apparatus with $|+X\rangle$ open and $|-X\rangle$ blocked.
(a) What is the probability that a photon in this state will pass through an ideal polarizer with its transmission axis oriented in the $|Y\rangle$ direction? [5 pts]
(b) What is the probability that a photon in this state will pass through an ideal polarizer with its transmission axis oriented in the $\left|Y^{\prime}\right\rangle$ making an angle $\phi$ with the $|Y\rangle$ axis? [5 pts]
(c) Determine the state $|\psi\rangle$ in the $|R\rangle$ and $|L\rangle$ basis. [5 pts]
(d) A beam carrying $N$ photons per second, each in the state $|\psi\rangle$, is totally absorbed by a black disk with its normal to the surface in the z direction. How large is the torque,

$$
\vec{\tau}=\frac{d \vec{J}}{d t}
$$

exerted on the disk by the photons with
(i) right circular polarization (photons in the state $|R\rangle=\frac{1}{\sqrt{2}}(|X\rangle+i|Y\rangle)$ ) [5 pts]
(ii) left circular polarization (photons in the state $|L\rangle=\frac{1}{\sqrt{2}}(|X\rangle-i|Y\rangle)$ ) [5 pts]

Note: The magnitude of the angular momentum of a photon is $\hbar$.
(e) Suppose the photons described by the state $|\psi\rangle$ are produced from a linearly polarized light (at an angle of $45^{\circ}$ to the $x$ and $y$ axes) of wavelength $5890 \AA$ incident normally on a birefringent crystal that has its optic axis parallel to the face of the crystal along the x-axis If the index of refraction for light of this wavelength polarized along $y$ (perpendicular to the optic axis) is 1.66 and the index of refraction for the light polarized along $x$ (parallel to the optic axis) is 1.49 , what should be the length of the crystal that produces photons in the state $|\psi\rangle$.
3. For a spin half particle described by the state

$$
|+n\rangle=\cos \left(\frac{\theta}{2}\right)|+Z\rangle+e^{i \varphi} \sin \left(\frac{\theta}{2}\right)|-Z\rangle
$$

(a) Express the state vector in $S_{y}$ basis and $S_{x}$ basis [5 pts]
(b) Find the probability amplitudes that the measurement for $S_{y}$ yields $+\hbar / 2$ and $-\hbar / 2$ ? [5 pts]
(c) What are the probabilities that the measurement for $S_{y}$ yields $+\hbar / 2$ and $-\hbar / 2$ ? [ 5 pts ]
(d) Find the expectation values $\left\langle\hat{S}_{y}\right\rangle,\left\langle\hat{S}_{y}^{2}\right\rangle$, and the uncertainty $\Delta S_{y}$. [10 pts]
4. The general uncertainty principle states that for two physical observables represented by operators $\hat{A}$ and $\hat{B}$, the uncertainties in simultaneous measurement of these physical observables is determined by the inequality

$$
(\Delta A)^{2}(\Delta B)^{2} \geq \frac{1}{4}|\langle\hat{C}\rangle|^{2},
$$

where the operator $\hat{C}$ is related by the commutation relation of the two operators given by

$$
[\hat{A}, \hat{B}]=i \hat{C} .
$$

(a) Using the commutation relations for the components of the angular momentum operators

$$
\left[\hat{J}_{x}, \hat{J}_{y}\right]=i \hbar \hat{J}_{z},\left[\hat{J}_{y}, \hat{J}_{z}\right]=i \hbar \hat{J}_{x},\left[\hat{J}_{z}, \hat{J}_{x}\right]=i \hbar \hat{J}_{y}
$$

determine the uncertainty relations for simultaneous measurements of any two components of the angular momentum. [10 pts]
(b) Suppose the operator $\hat{A}$ is the x -component for the position operator, $\hat{x}$, and $\hat{B}$ is the x -component for momentum operator, $\hat{p}_{x}$, given by

$$
\hat{p}_{x}=-i \hbar \frac{d}{d x}, \hat{x}=x
$$

by deriving the commutation relation for

$$
[\hat{A}, \hat{B}]=i \hat{C} .
$$

find the Heisenberg uncertainty relation for momentum and position. [15 pts]
5. Using the commutation relation for $\hat{p}_{x}$ and $\hat{x}$ you determined in the previous problem, for the operators $\hat{a}$ and $\hat{a}^{\dagger}$, defined by

$$
\begin{aligned}
\hat{a} & =\sqrt{\frac{m \omega}{2 \hbar}} \hat{x}+i \frac{1}{\sqrt{2 m \omega \hbar}} \hat{p} \\
\hat{a}^{\dagger} & =\sqrt{\frac{m \omega}{2 \hbar}} \hat{x}-i \frac{1}{\sqrt{2 m \omega \hbar}} \hat{p}
\end{aligned}
$$

derive the commutation relation $\left[\hat{a}, \hat{a}^{\dagger}\right] .[20 \mathrm{pts}]$

## Part II: Take-home Problems

1. A spin half particle is described by the state vector

$$
|\psi\rangle=\frac{1}{2}|+X\rangle+\frac{i \sqrt{3}}{2}|-X\rangle
$$

(a) Find the matrix representation of the state vector $|\psi\rangle$ in the $J_{x}$-basis and using matrix mechanics show that $|\psi\rangle$ is properly normalized. [5 pts]
(b) By directly using $| \pm Z\rangle$ expressed in terms of $| \pm X\rangle$, find the matrix representation of the operator $\hat{J}_{x}$ in z-basis ( $J_{z}$ basis). [5 pts]
(c) Determine the transformation matrix that changes the matrix representation for the operator $\hat{J}_{x}$ in z-basis ( $J_{z}$ basis) to a matrix representation in x-basis ( $J_{x}$ basis). [5 pts]
(d) Using matrices only find expectation values $\left\langle\hat{J}_{z}\right\rangle,\left\langle\hat{J}_{z}^{2}\right\rangle$, and the uncertainty $\Delta J_{z}$. [10 pts]
2. A one dimensional quantum harmonic oscillator can be described by the operators $\left(\hat{a}^{\dagger}, \hat{a}\right)$ known as the Ladder operators. These operators are related to position $(\hat{x})$ and momentum $(\hat{p})$ operators by

$$
\begin{aligned}
\hat{a} & =\sqrt{\frac{m \omega}{2 \hbar}} \hat{x}+i \frac{1}{\sqrt{2 m \omega \hbar}} \hat{p} \\
\hat{a}^{\dagger} & =\sqrt{\frac{m \omega}{2 \hbar}} \hat{x}-i \frac{1}{\sqrt{2 m \omega \hbar}} \hat{p}
\end{aligned}
$$

where $m$ and $\omega$ are real constants. Suppose the energy of a quantum harmonic oscillator is described by the energy operator $\hat{H}=\hbar \omega\left(\hat{a}^{\dagger} \hat{a}+\frac{1}{2}\right)$. Let's assume that there are only two energy eigenstates $|0\rangle$ (the ground state) and $|1\rangle$ (the excited state) with corresponding eigen values $\frac{\hbar \omega}{2}$ and $\frac{3 \hbar \omega}{2}$, respectively. That means

$$
\hat{H}|0\rangle=\frac{\hbar \omega}{2}|0\rangle, \hat{H}|1\rangle=\frac{3 \hbar \omega}{2}|1\rangle
$$

The operator $\hat{a}$ lowers and $\hat{a}^{\dagger}$ raises the state by one like the angular momentum lowering and raising operators $\hat{J}_{-}$and $\hat{J}_{+}$we studied in class. This means when these operators act on the two eigenstates, it gives the following:

$$
\begin{aligned}
\hat{a}|0\rangle & =0, & & \hat{a}|1\rangle=1|0\rangle \\
\hat{a}^{\dagger}|0\rangle & =1|1\rangle, & & \hat{a}^{\dagger}|1\rangle=0
\end{aligned}
$$

Note: The eigenstates $|0\rangle$ and $|1\rangle$ form a complete orthonormal set of vectors.
(a) Express the position operator, $\hat{x}$, in terms of the ladder operators $\left(\hat{a}^{\dagger}, \hat{a}\right)$. [3 pts]
(b) Find the matrix representation of the energy $\hat{H}$ and the position $\hat{x}$ operators (using the result in (a)) in the $|0\rangle$ and $|1\rangle$ basis.[5 pts]

$$
\begin{aligned}
& \hat{H} \xrightarrow{|0\rangle \text { and }|1\rangle \operatorname{basis}}\left(\begin{array}{ll}
\langle 0| \hat{H}|1\rangle & \langle 0| \hat{H}|1\rangle \\
\langle 1| \hat{H}|0\rangle & \langle 1| \hat{H}|1\rangle
\end{array}\right) \\
& \stackrel{x}{|0\rangle \text { and }|1\rangle \operatorname{basis}}\left(\begin{array}{ll}
\langle 0| \hat{x}|1\rangle & \langle 0| \hat{x}|1\rangle \\
\langle 1| \hat{x}|0\rangle & \langle 1| \hat{x}|1\rangle
\end{array}\right)
\end{aligned}
$$

(c) Determine the eigenvalues for the position operator $\hat{x}$ and show that the corresponding eigen vectors are given by [ 7 pts ]

$$
\begin{aligned}
\left|x_{1}\right\rangle & =\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \\
\left|x_{2}\right\rangle & =\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)
\end{aligned}
$$

(d) The momentum operator $\hat{p}$ eigen states are found to be

$$
\left|p_{1}\right\rangle=\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle) \text { and }\left|p_{2}\right\rangle=\frac{1}{\sqrt{2}}(|0\rangle-i|1\rangle)
$$

Express these eigenstates in the $\left|x_{1}\right\rangle$ and $\left|x_{2}\right\rangle$ basis. [5 pts]
(e) Determine the matrix representation of the energy operator $\hat{H}$ in the $x$ basis. [5 pts]

