PHYS 4380 Midterm Exam

October 11, 2018

Instructor: Dr. Daniel Erenso Name:———————————

Instructions and important notes

- This is Part I of the midterm exam and it consists of three problems worth 50 points. Problems 4 and 5 in Part II are the take-home part of the exam.
- Please pay attention to italicized or bold phrases.
- To receive full credit, your work must be clear and complete.
- Begin the solution of each problem on a new page. Do not use the back pages!
- The solutions to each problems must be presented in order.
- Please box in the final result to each part of the problems when it is appropriate.
- You must attach all the pages of this exam on top of the pages of your properly ordered solutions.

YOU HAVE 85 MINUTES TO COMPLETE THIS TEST

	Part I			Part II		
Problem	1	2	3	4	5	Total
Score	/20	/20	/10	/25	/25	/100

Part I **In-class**

1. Provide a short and brief answer

(a) Consider the vector in a complex Cartesian vector space

$$\vec{A} = 3\hat{y} - 4\hat{z},$$

Suppose the unit vectors $\hat{x}, \hat{y}$, and $\hat{z}$ can be represented by $|e_1\rangle, |e_2\rangle$, and $|e_3\rangle$. Express $\vec{A}$ and $\vec{A}^*$ using Dirac notation [3 pts]

Solution:

$$|A\rangle = 3\,|e_2\rangle - 4\,|e_3\rangle\,, \Rightarrow |\langle A|\rangle = 3\,\langle e_2| - 4\,\langle e_3|$$

(b) Consider the two SG devices shown in the figure. The first device (SG-1) has none uniform magnetic field in the x-direction and the second device (SG-2) has a non uniform magnetic field in the z-direction. A spin-half

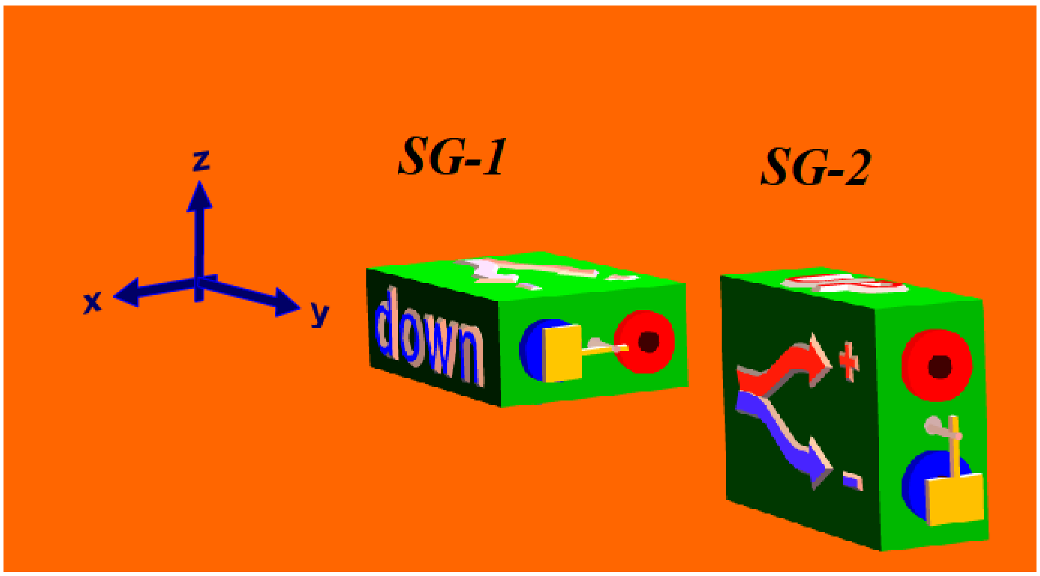

Figure 1: Two SG devices. SG-1 magnetic field gradient is in the x-direction and for SG-2 the magnetic field gradient is in the z-direction.

particle in a state $|+X\rangle$ is incident on SG-1.

(i) What is the probability of this particle exiting SG-1 in the state $|+X\rangle$.[2 pts]

Sol: It is one since the magnetic gradient is also in the x direction.

(ii) What is the probability of this particle exiting SG-1 in the state $|-X\rangle$.[2 pts]

Sol: It is zero since the magnetic gradient is also in the x direction and can no change the state.

(iii.) Suppose the particle exiting the SG-1 in a $|+X\rangle$ state enters SG-2. What is the probability that this particle exiting SG-2 in a state $|-Z\rangle$. [2 pts]

Sol: It is 0.5 (50%) since the magnetic gradient is z direction that could change the state to $|+Z\rangle$ and $|-Z\rangle$.

(iV) Suppose you rotated SG-2 about the y axis by $\pi/2$, what would be the probability that the particle exiting SG-2 is in a state $|-X\rangle$. [2 pts]

Sol: It would be zero.

(c) Suppose there are two particles of same type. Particle one is in a state described by the ket vector

$$|\psi_1\rangle = c_1\,|a_1\rangle + c_2\,|a_2\rangle$$

and particle two in a state described by the ket vector

$$|\psi_2\rangle = e^{i\theta} c_1 |a_1\rangle + e^{i\theta} c_2 |a_2\rangle ,$$

where the vectors $\{|a_1\rangle , |a_2\rangle\}$ form an orthonormal set of basis vectors, c_1 and c_2 are complex constants and θ is a real constant. Let the operator $\hat{A}$ represent some measurable physical observable such that

$$\hat{A} |a_1\rangle = A_1 |a_1\rangle , \hat{A} |a_2\rangle = A_2 |a_2\rangle .$$

Show that

$$\langle\psi_2| \hat{A} |\psi_2\rangle = \langle\psi_1| \hat{A} |\psi_1\rangle .$$

[4 pts]

Sol: Using the given state vector, we have

$$\begin{aligned} |\psi_2\rangle &= e^{i\theta} (c_1 |a_1\rangle + c_2 |a_2\rangle) = e^{i\theta} |\psi_1\rangle \\ \Rightarrow & \langle\psi_2| = e^{-i\theta} c_1^* |a_1\rangle + e^{-i\theta} c_2^* |a_2\rangle = e^{-i\theta} \langle\psi_1| \end{aligned}$$

so that

$$\langle\psi_2| \hat{A} |\psi_2\rangle = e^{-i\theta} \langle\psi_1| \hat{A} e^{i\theta} |\psi_1\rangle = \langle\psi_1| \hat{A} |\psi_1\rangle .$$

(d) Explain briefly the similarities and differences of the two virtual SG experiments we discussed in class and shown in the figures below. [5 pts]

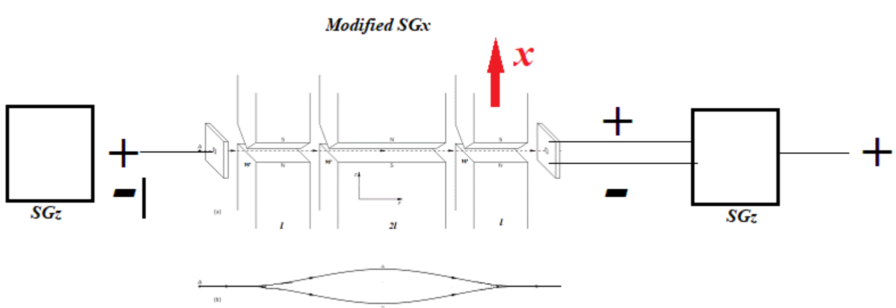

Figure 2: Modified SGx device with both states ($|+X\rangle$ and $|-X\rangle$) open.

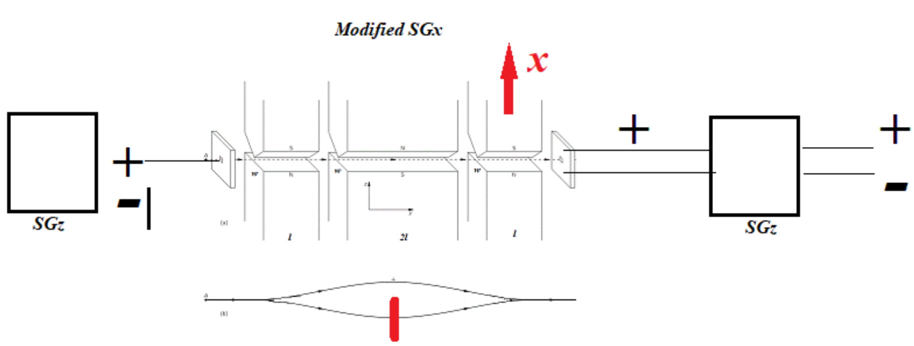

Figure 3: Modified SGx apparatus with $|+X\rangle$ open and $|-X\rangle$ blocked.

Sol: Refer to the note

2.

I. The polarization state for a photon propagating in the z-direction is given by

$$|\psi\rangle = -\frac{i}{\sqrt{3}}|X\rangle + \sqrt{\frac{2}{3}}|Y\rangle .$$

Without using matrices,

(a) Determine the state $|\psi\rangle$ in the $|R\rangle$ and $|L\rangle$ basis. [5 pts]

Sol: using the completeness relation, one can write

$$\begin{aligned}|\psi\rangle &= (|R\rangle\langle R| + |L\rangle\langle L|)\left(-\frac{i}{\sqrt{3}}|X\rangle + \sqrt{\frac{2}{3}}|Y\rangle\right) \\ &= -\frac{i}{\sqrt{3}}(|R\rangle\langle R|X\rangle + |L\rangle\langle L|X\rangle) + \sqrt{\frac{2}{3}}(|R\rangle\langle R|Y\rangle + |L\rangle\langle L|Y\rangle).\end{aligned}$$

so that using

$$\begin{aligned}|R\rangle &= \frac{1}{\sqrt{2}}(|X\rangle + i|Y\rangle), |L\rangle = \frac{1}{\sqrt{2}}(|X\rangle - i|Y\rangle) \\ &\Rightarrow \langle R| = \frac{1}{\sqrt{2}}(\langle X| - i\langle Y|), \langle L| = \frac{1}{\sqrt{2}}(\langle X| + i\langle Y|)\end{aligned}$$

one finds

$$\begin{aligned}|\psi\rangle &= (|R\rangle\langle R| + |L\rangle\langle L|)\left(-\frac{i}{\sqrt{3}}|X\rangle + \sqrt{\frac{2}{3}}|Y\rangle\right) \\ &= -\frac{i}{\sqrt{6}}(|R\rangle + |L\rangle) - \frac{i}{\sqrt{3}}(|R\rangle - |L\rangle) \\ &= -\left(\frac{i}{\sqrt{3}} + \frac{i}{\sqrt{6}}\right)|R\rangle + \left(\frac{i}{\sqrt{3}} - \frac{i}{\sqrt{6}}\right)|L\rangle .\end{aligned}$$

or

$$|\psi\rangle = c_R|R\rangle + c_L|L\rangle .$$

where

$$c_R = -\left(\frac{i}{\sqrt{3}} + \frac{i}{\sqrt{6}}\right), c_L = \left(\frac{i}{\sqrt{3}} - \frac{i}{\sqrt{6}}\right)$$

(b) Suppose 36000 photons each in a state $|\psi\rangle$ are incident on a black disk in one hour at a uniform rate. If all the incident photons are totally absorbed by the disk with its normal to the surface in the z direction. Find the magnitude of the net torque on the disk,

$$\vec{\tau} = \frac{d\vec{J}}{dt}.$$

[7.5 pts]

Sol: The net angular momentum

$$\begin{aligned}J &= J_R - J_L = (P_R - P_R)\hbar = \left(|c_R|^2 - |c_L|^2\right)\hbar = \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)^2 - \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)^2 \\ &\Rightarrow J = \frac{4}{\sqrt{18}}\hbar = \frac{4}{3\sqrt{2}}\hbar = \frac{2\sqrt{2}}{3}\hbar = 0.94\hbar\end{aligned}$$

Then the torque

$$\vec{\tau} = \frac{N}{t}J = \frac{36000}{3600}\frac{4}{\sqrt{18}}\hbar = \frac{40}{\sqrt{18}}\hbar = 9.4\hbar.$$

becomes

Note: The magnitude of the angular momentum of a photon is $\hbar$.

II. For a spin half particle in a the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|+Z\rangle + e^{i\varphi}|-Z\rangle\right],$$

without using matrices,

(a) express the state vector in S_x basis [5 pts]

Sol: using the completeness relation, one can write

$$\begin{aligned}|\psi\rangle &= \left(|+X\rangle\langle +X| + |-X\rangle\langle -X|\right)\frac{1}{\sqrt{2}}\left[|+Z\rangle + e^{i\varphi}|-Z\rangle\right]\\ &= \frac{1}{\sqrt{2}}\Big(\left(|+X\rangle\langle +X|+Z\rangle + |-X\rangle\langle -X|+Z\rangle\right) + e^{i\varphi}\left(|+X\rangle\langle +X|-Z\rangle + |-X\rangle\langle -X|-Z\rangle\right)\Big)\end{aligned}$$

so that using

$$|\pm X\rangle = \frac{1}{\sqrt{2}}\left(|+Z\rangle \pm |-Z\rangle\right) \Rightarrow \langle \pm X| = \frac{1}{\sqrt{2}}\left(\langle +Z| \pm \langle -Z|\right),$$

we find

$$\begin{aligned}|\psi\rangle &= \frac{1}{2}\left(\left(|+X\rangle + |-X\rangle\right) + e^{i\varphi}\left(|+X\rangle - |-X\rangle\right)\right)\\ &= \frac{\left(1+e^{i\varphi}\right)}{2}|+X\rangle + \frac{\left(1-e^{i\varphi}\right)}{2}|-X\rangle\end{aligned}$$

or

$$|\psi\rangle = c_+|+X\rangle + \frac{\left(1-e^{i\varphi}\right)}{2}|-X\rangle$$

where

$$c_+ = \frac{\left(1+e^{i\varphi}\right)}{2}, c_- = \frac{\left(1-e^{i\varphi}\right)}{2}$$

(b) find the expectation values $\left\langle \hat{S}_x \right\rangle$, $\left\langle \hat{S}_x^2 \right\rangle$, and the uncertainty ΔS_x. [7.5 pts]

Sol:

$$\begin{aligned}\left\langle \hat{S}_x \right\rangle &= |c_+|^2\left(\frac{\hbar}{2}\right) + |c_-|^2\left(-\frac{\hbar}{2}\right) =\\ \left\langle \hat{S}_x^2 \right\rangle &= |c_+|^2\left(\frac{\hbar}{2}\right)^2 + |c_-|^2\left(-\frac{\hbar}{2}\right)^2,\\ \Delta S_x &= \sqrt{\left\langle \hat{S}_x^2 \right\rangle - \left\langle \hat{S}_x \right\rangle^2}\end{aligned}$$

Noting that

$$\begin{aligned}|c_+|^2 &= \frac{\left(1+e^{i\varphi}\right)}{2}\frac{\left(1+e^{-i\varphi}\right)}{2} = \frac{\left(2+e^{i\varphi}+e^{-i\varphi}\right)}{4} = \frac{\left(2+2\cos(\varphi)\right)}{4} = \frac{1+\cos(\varphi)}{2}\\ |c_-|^2 &= \frac{\left(1-e^{i\varphi}\right)}{2}\frac{\left(1-e^{-i\varphi}\right)}{2} = \frac{\left(2-e^{i\varphi}-e^{-i\varphi}\right)}{4} = \frac{\left(2-2\cos(\varphi)\right)}{4} = \frac{1-\cos(\varphi)}{2}\end{aligned}$$

we find

$$\begin{aligned}\left\langle \hat{S}_x \right\rangle &= \frac{\hbar}{2}\cos(\varphi)\\ \left\langle \hat{S}_x^2 \right\rangle &= \frac{\hbar^2}{4},\\ \Delta S_x &= \sqrt{\frac{\hbar^2}{4}\left(1-\cos^2(\varphi)\right)} = \frac{\hbar}{2}\sin(\varphi)\end{aligned}$$

3. **(a)** Suppose the operator describing the y-component for the position of a particle is, $\hat{y}$, and the operator for the y-component of the momentum this particle is $\hat{p}_y$.The operators are given by

$$\hat{p}_y = -i\hbar\frac{d}{dy}, \hat{y} = y$$

derive the commutation relation for these two operators, $[\hat{y}, \hat{p}_y]$ .and find the Heisenberg uncertainty relation for momentum and position. [5 pts]

(b) Using the commutation relation for $\hat{p}_y$ and $\hat{y}$ you determined in the previous problem, for the operators $\hat{a}$ and $\hat{a}^\dagger$, defined by

$$\begin{aligned}\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\hat{y} + i\frac{1}{\sqrt{2m\omega\hbar}}\hat{p}_y \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\hat{y} - i\frac{1}{\sqrt{2m\omega\hbar}}\hat{p}_y\end{aligned}$$

derive the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right]$. [5 pts]

Part II: Take-home

4. A spin half particle is described by the state vector

$$|\psi\rangle = \frac{1}{2}|+X\rangle + \frac{i\sqrt{3}}{2}|-X\rangle$$

(a) Find the matrix representation of the state vector $|\psi\rangle$ in the J_x-basis and using matrix mechanics show that $|\psi\rangle$ is properly normalized. [5 pts]

(b) By directly using $|\pm Z\rangle$ expressed in terms of $|\pm X\rangle$, find the matrix representation of the operator $\hat{J}_x$ in z-basis (J_z basis).[5 pts]

(c) Determine the transformation matrix that changes the matrix representation for the operator $\hat{J}_x$ in z-basis (J_z basis) to a matrix representation in x-basis (J_x basis). [5 pts]

(d) Using matrices only find expectation values $\left\langle \hat{J}_z \right\rangle$, $\left\langle \hat{J}_z^2 \right\rangle$, and the uncertainty ΔJ_z. [10 pts]

Sol: **(a)** The matrix representation of the state vector $|\psi\rangle$ in the J_x-basis

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \langle +X|\,\psi\rangle \\ \langle -X|\,\psi\rangle \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -i\sqrt{3} \end{bmatrix}$$

(b) Using

$$|\pm Z\rangle = \frac{1}{\sqrt{2}}\left(|+X\rangle \pm |-X\rangle\right)$$

the matrix representation of the operator $\hat{J}_x$ in z-basis (J_z basis).

$$\begin{aligned}
&\begin{bmatrix} \langle +Z|\,\hat{J}_x\,|+Z\rangle & \langle +Z|\,\hat{J}_x\,|-Z\rangle \\ \langle -Z|\,\hat{J}_x\,|+Z\rangle & \langle -Z|\,\hat{J}_x\,|-Z\rangle \end{bmatrix} \\
= \; & \frac{1}{2}\begin{bmatrix} \left(\langle +X| + \langle -X|\right)\hat{J}_x\left(|+X\rangle + |-X\rangle\right) & \left(\langle +X| + \langle -X|\right)\hat{J}_x\left(|+X\rangle - |-X\rangle\right) \\ \left(\langle +X| - \langle -X|\right)\hat{J}_x\left(|+X\rangle + |-X\rangle\right) & \left(\langle +X| - \langle -X|\right)\hat{J}_x\left(|+X\rangle - |-X\rangle\right) \end{bmatrix} \\
\Rightarrow \; & \begin{bmatrix} \langle +Z|\,\hat{J}_x\,|+Z\rangle & \langle +Z|\,\hat{J}_x\,|-Z\rangle \\ \langle -Z|\,\hat{J}_x\,|+Z\rangle & \langle -Z|\,\hat{J}_x\,|-Z\rangle \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\hbar}{2} - \frac{\hbar}{2} & \frac{\hbar}{2} + \frac{\hbar}{2} \\ \frac{\hbar}{2} + \frac{\hbar}{2} & \frac{\hbar}{2} - \frac{\hbar}{2} \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

(c) Using the relation we determined in class

$$A' = T^\dagger A T$$

where

$$\begin{aligned}
A' &= \begin{bmatrix} \langle b_1|\,\hat{A}\,|b_1\rangle & \langle b_1|\,\hat{A}\,|b_2\rangle \\ \langle b_2|\,\hat{A}\,|b_1\rangle & \langle b_2|\,\hat{A}\,|b_2\rangle \end{bmatrix}, A = \begin{bmatrix} \langle a_1|\,\hat{A}\,|a_1\rangle & \langle a_1|\,\hat{A}\,|a_2\rangle \\ \langle a_2|\,\hat{A}\,|a_1\rangle & \langle a_2|\,\hat{A}\,|a_2\rangle \end{bmatrix}, \\
T &= \begin{bmatrix} \langle a_1|\,b_1\rangle & \langle a_1|\,b_2\rangle \\ \langle a_2|\,b_1\rangle & \langle a_2|\,b_2\rangle \end{bmatrix}.
\end{aligned}$$

the transformation matrix, T, that changes the matrix representation for the operator $\hat{J}_x$ in z-basis (J_z basis$\Rightarrow |a_1\rangle, |a_2\rangle$) to a matrix representation in x-basis (J_x basis$\Rightarrow |b_1\rangle, |b_2\rangle$) can be expressed as

$$T = \begin{bmatrix} \langle a_1|\,b_1\rangle & \langle a_1|\,b_2\rangle \\ \langle a_2|\,b_1\rangle & \langle a_2|\,b_2\rangle \end{bmatrix} = \begin{bmatrix} \langle +Z|\,+X\rangle & \langle +Z|\,-X\rangle \\ \langle -Z|\,+X\rangle & \langle -Z|\,-X\rangle \end{bmatrix}$$

so that using

$$|\pm X\rangle = \frac{1}{\sqrt{2}}\left(|+Z\rangle \pm |-Z\rangle\right)$$

we find

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(d) Noting that in the $\hat{J}_z$ basis

$$\begin{aligned}
\left\langle \hat{J}_z \right\rangle &= \begin{pmatrix} \langle\psi| +Z\rangle & \langle\psi| -Z\rangle \end{pmatrix} \begin{bmatrix} \langle +Z|\, \hat{J}_z \,|+Z\rangle & \langle +Z|\, \hat{J}_z \,|-Z\rangle \\ \langle -Z|\, \hat{J}_z \,|+Z\rangle & \langle -Z|\, \hat{J}_z \,|-Z\rangle \end{bmatrix} \begin{bmatrix} \langle +Z|\, \psi\rangle \\ \langle -Z|\, \psi\rangle \end{bmatrix} \\
\left\langle \hat{J}_z^2 \right\rangle &= \begin{pmatrix} \langle\psi| +Z\rangle & \langle\psi| -Z\rangle \end{pmatrix} \begin{bmatrix} \langle +Z|\, \hat{J}_z^2 \,|+Z\rangle & \langle +Z|\, \hat{J}_z^2 \,|-Z\rangle \\ \langle -Z|\, \hat{J}_z^2 \,|+Z\rangle & \langle -Z|\, \hat{J}_z^2 \,|-Z\rangle \end{bmatrix} \begin{bmatrix} \langle +Z|\, \psi\rangle \\ \langle -Z|\, \psi\rangle \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{bmatrix} \langle +Z|\, \psi\rangle \\ \langle -Z|\, \psi\rangle \end{bmatrix} &= \begin{bmatrix} \langle +Z| \left(\frac{1}{2}\,|+X\rangle + \frac{i\sqrt{3}}{2}\,|-X\rangle\right) \\ \langle -Z| \left(\frac{1}{2}\,|+X\rangle + \frac{i\sqrt{3}}{2}\,|-X\rangle\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2} \end{bmatrix} \\
&\Rightarrow \begin{pmatrix} \langle\psi| +Z\rangle & \langle\psi| -Z\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \\
\begin{bmatrix} \langle +Z|\, \hat{J}_z \,|+Z\rangle & \langle +Z|\, \hat{J}_z \,|-Z\rangle \\ \langle -Z|\, \hat{J}_z \,|+Z\rangle & \langle -Z|\, \hat{J}_z \,|-Z\rangle \end{bmatrix} &= \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
\begin{bmatrix} \langle +Z|\, \hat{J}_z^2 \,|+Z\rangle & \langle +Z|\, \hat{J}_z^2 \,|-Z\rangle \\ \langle -Z|\, \hat{J}_z^2 \,|+Z\rangle & \langle -Z|\, \hat{J}_z^2 \,|-Z\rangle \end{bmatrix} &= \frac{\hbar^2}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

we have

$$\begin{aligned}
\left\langle \hat{J}_z \right\rangle &= \begin{pmatrix} \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2} \end{bmatrix} \\
&\Rightarrow \left\langle \hat{J}_z \right\rangle = \frac{\hbar}{2} \begin{pmatrix} \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2} \end{bmatrix} \\
&\Rightarrow \left\langle \hat{J}_z \right\rangle = \frac{\hbar}{2}\left(\frac{1}{8} + \frac{3}{8}\right) - \frac{\hbar}{2}\left(\frac{1}{8} + \frac{3}{8}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \hat{J}_z^2 \right\rangle &= \begin{pmatrix} \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \frac{\hbar^2}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2} \end{bmatrix} \\
&\Rightarrow \left\langle \hat{J}_z \right\rangle = \frac{\hbar^2}{4} \begin{pmatrix} \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} + \frac{i\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} - \frac{i\sqrt{3}}{2} \end{bmatrix} \\
&\Rightarrow \left\langle \hat{J}_z \right\rangle = \frac{\hbar^2}{4}\left(\frac{1}{8} + \frac{3}{8}\right) + \frac{\hbar^2}{4}\left(\frac{1}{8} + \frac{3}{8}\right) = \frac{\hbar^2}{4}
\end{aligned}$$

so that

$$\Delta J_z = \sqrt{\left\langle \hat{J}_z^2 \right\rangle - \left\langle \hat{J}_z \right\rangle^2} = \frac{\hbar}{2}$$

5. A one dimensional quantum harmonic oscillator can be described by the operators $(\hat{a}^\dagger, \hat{a})$ known as the Ladder operators. These operators are related to position $(\hat{x})$ and momentum $(\hat{p})$ operators by

$$\begin{aligned}
\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{1}{\sqrt{2m\omega\hbar}}\hat{p}, \\
\hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{1}{\sqrt{2m\omega\hbar}}\hat{p},
\end{aligned}$$

where m and ω are real constants. Suppose the energy of a quantum harmonic oscillator is described by the energy operator $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. Let's assume that there are only two energy eigenstates $|0\rangle$ (the ground state) and $|1\rangle$ (the excited state) with corresponding eigen values $\frac{\hbar\omega}{2}$ and $\frac{3\hbar\omega}{2}$, respectively. That means

$$\hat{H}\,|0\rangle = \frac{\hbar\omega}{2}\,|0\rangle\,, \hat{H}\,|1\rangle = \frac{3\hbar\omega}{2}\,|1\rangle$$

The operator $\hat{a}$ lowers and $\hat{a}^\dagger$ raises the state by one like the angular momentum lowering and raising operators $\hat{J}_-$ and $\hat{J}_+$ we studied in class. This means when these operators act on the two eigenstates, it gives the following:

$$\begin{aligned} \hat{a}\,|0\rangle &= 0, & \hat{a}\,|1\rangle = 1\,|0\rangle\,, \\ \hat{a}^\dagger\,|0\rangle &= 1\,|1\rangle\,, & \hat{a}^\dagger\,|1\rangle = 0. \end{aligned}$$

Note: The eigenstates $|0\rangle$ and $|1\rangle$ form a complete orthonormal set of vectors.

(a) Express the position operator, $\hat{x}$, in terms of the ladder operators $(\hat{a}^\dagger, \hat{a})$. [3 pts]

(b) Find the matrix representation of the energy $\hat{H}$ and the position $\hat{x}$ operators (using the result in (a)) in the $|0\rangle$ and $|1\rangle$ basis.[5 pts]

$$\hat{H} \xrightarrow{\ |0\rangle \text{ and } |1\rangle \text{ basis}\ } \begin{pmatrix} \langle 0|\,\hat{H}\,|1\rangle & \langle 0|\,\hat{H}\,|1\rangle \\ \langle 1|\,\hat{H}\,|0\rangle & \langle 1|\,\hat{H}\,|1\rangle \end{pmatrix}$$

$$\hat{x} \xrightarrow{\ |0\rangle \text{ and } |1\rangle \text{ basis}\ } \begin{pmatrix} \langle 0|\,\hat{x}\,|1\rangle & \langle 0|\,\hat{x}\,|1\rangle \\ \langle 1|\,\hat{x}\,|0\rangle & \langle 1|\,\hat{x}\,|1\rangle \end{pmatrix}$$

(c) Determine the eigenvalues for the position operator $\hat{x}$ and show that the corresponding eigen vectors are given by [7 pts]

$$\begin{aligned} |x_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ |x_2\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{aligned}$$

(d) The momentum operator $\hat{p}$ eigen states are found to be

$$|p_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right) \text{ and } |p_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i\,|1\rangle\right).$$

Express these eigenstates in the $|x_1\rangle$ and $|x_2\rangle$ basis. [5 pts]

(e) Determine the matrix representation of the energy operator $\hat{H}$ in the x basis. [5 pts]

Sol: **(a)** The position operator, $\hat{x}$, in terms of the ladder operators $(\hat{a}^\dagger, \hat{a})$ can be expressed as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$$

(b) Find the matrix representation of the energy $\hat{H}$ and the position $\hat{x}$ operators (using the result in (a)) in the $|0\rangle$ and $|1\rangle$ basis.[5 pts]

$$\begin{aligned} \begin{pmatrix} \langle 0|\,\hat{H}\,|0\rangle & \langle 0|\,\hat{H}\,|1\rangle \\ \langle 1|\,\hat{H}\,|0\rangle & \langle 1|\,\hat{H}\,|1\rangle \end{pmatrix} &= \begin{pmatrix} \langle 0|\,\frac{\hbar\omega}{2}\,|0\rangle & \langle 0|\,\frac{3\hbar\omega}{2}\,|1\rangle \\ \langle 1|\,\frac{\hbar\omega}{2}\,|0\rangle & \langle 1|\,\frac{3\hbar\omega}{2}\,|1\rangle \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \langle 0|\,\hat{H}\,|0\rangle & \langle 0|\,\hat{H}\,|1\rangle \\ \langle 1|\,\hat{H}\,|0\rangle & \langle 1|\,\hat{H}\,|1\rangle \end{pmatrix} &= \begin{pmatrix} \frac{\hbar\omega}{2}\,\langle 0|\,|0\rangle & \frac{3\hbar\omega}{2}\,\langle 0|\,|1\rangle \\ \frac{\hbar\omega}{2}\,\langle 1|\,|0\rangle & \frac{3\hbar\omega}{2}\,\langle 1|\,|1\rangle \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \langle 0|\,\hat{H}\,|0\rangle & \langle 0|\,\hat{H}\,|1\rangle \\ \langle 1|\,\hat{H}\,|0\rangle & \langle 1|\,\hat{H}\,|1\rangle \end{pmatrix} &= \begin{pmatrix} \frac{\hbar\omega}{2} & 0 \\ 0 & \frac{3\hbar\omega}{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} \langle 0|\,\hat{x}\,|0\rangle & \langle 0|\,\hat{x}\,|1\rangle \\ \langle 1|\,\hat{x}\,|0\rangle & \langle 1|\,\hat{x}\,|1\rangle \end{pmatrix} &= \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} \langle 0|\left(\hat{a} + \hat{a}^\dagger\right)|0\rangle & \langle 0|\left(\hat{a} + \hat{a}^\dagger\right)|1\rangle \\ \langle 1|\left(\hat{a} + \hat{a}^\dagger\right)|0\rangle & \langle 1|\left(\hat{a} + \hat{a}^\dagger\right)|1\rangle \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \langle 0|\,\hat{x}\,|0\rangle & \langle 0|\,\hat{x}\,|1\rangle \\ \langle 1|\,\hat{x}\,|0\rangle & \langle 1|\,\hat{x}\,|1\rangle \end{pmatrix} &= \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} \langle 0|\,\hat{a}^\dagger\,|0\rangle & \langle 0|\,\hat{a}\,|1\rangle \\ \langle 1|\,\hat{a}^\dagger\,|0\rangle & \langle 1|\,\hat{a}\,|1\rangle \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \langle 0|\,\hat{x}\,|0\rangle & \langle 0|\,\hat{x}\,|1\rangle \\ \langle 1|\,\hat{x}\,|0\rangle & \langle 1|\,\hat{x}\,|1\rangle \end{pmatrix} &= \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} \langle 0|\,|1\rangle & \langle 0|\,|0\rangle \\ \langle 1|\,|1\rangle & \langle 1|\,|0\rangle \end{pmatrix} = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

(c) The eigen values are determined from

$$\det\begin{vmatrix} -\lambda & \sqrt{\frac{\hbar}{2m\omega}} \\ \sqrt{\frac{\hbar}{2m\omega}} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{\hbar}{2m\omega} = 0$$

$$\Rightarrow \lambda_1 = \sqrt{\frac{\hbar}{2m\omega}}, \lambda_2 = -\sqrt{\frac{\hbar}{2m\omega}}$$

The corresponding eigen vectors, for $\lambda_1 = \sqrt{\frac{\hbar}{2m\omega}}$

$$\begin{bmatrix} -\sqrt{\frac{\hbar}{2m\omega}} & \sqrt{\frac{\hbar}{2m\omega}} \\ \sqrt{\frac{\hbar}{2m\omega}} & -\sqrt{\frac{\hbar}{2m\omega}} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_1 = a_2$$

$$\Rightarrow |x_1\rangle = a_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

using the normalization condition

$$|x_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Similarly for $\lambda_1 = -\sqrt{\frac{\hbar}{2m\omega}}$

$$\begin{bmatrix} \sqrt{\frac{\hbar}{2m\omega}} & \sqrt{\frac{\hbar}{2m\omega}} \\ \sqrt{\frac{\hbar}{2m\omega}} & \sqrt{\frac{\hbar}{2m\omega}} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_2 = -a_1$$

$$\Rightarrow |x_2\rangle = a_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

using the normalization condition

$$|x_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

(d) The momentum operator $\hat{p}$ eigen states are found to be

$$|p_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \text{ and } |p_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle).$$

Using the completeness relation for the position eigenstates, one can write

$$\begin{aligned} |p_1\rangle &= (|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2|)\,|p_1\rangle = \langle x_1|\,p_1\rangle\,|x_1\rangle + \langle x_2|\,p_2\rangle\,|x_2\rangle \\ |p_2\rangle &= (|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2|)\,|p_2\rangle = \langle x_1|\,p_2\rangle\,|x_1\rangle + \langle x_2|\,p_2\rangle\,|x_2\rangle \end{aligned}$$

so that using

$$\begin{aligned} \langle x_1|\,p_1\rangle &= (\langle 0| + \langle 1|)\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) = \frac{1}{2}(1+i) \\ \langle x_2|\,p_1\rangle &= (\langle 0| - \langle 1|)\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) = \frac{1}{2}(1-i) \\ \langle x_1|\,p_2\rangle &= (\langle 0| + \langle 1|)\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) = \frac{1}{2}(1-i) \\ \langle x_2|\,p_2\rangle &= (\langle 0| - \langle 1|)\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) = \frac{1}{2}(1+i) \end{aligned}$$

we find

$$\begin{aligned} |p_1\rangle &= \frac{1}{2}[(1+i)\,|x_1\rangle + (1-i)\,|x_2\rangle], \\ |p_2\rangle &= \frac{1}{2}[(1-i)\,|x_1\rangle + (1+i)\,|x_2\rangle]. \end{aligned}$$

(e) Using the relation we derived in class

$$A' = T^\dagger A T$$

where

$$\begin{aligned} A' &= \begin{bmatrix} \langle b_1|\hat{A}|b_1\rangle & \langle b_1|\hat{A}|b_2\rangle \\ \langle b_2|\hat{A}|b_1\rangle & \langle b_2|\hat{A}|b_2\rangle \end{bmatrix}, A = \begin{bmatrix} \langle a_1|\hat{A}|a_1\rangle & \langle a_1|\hat{A}|a_2\rangle \\ \langle a_2|\hat{A}|a_1\rangle & \langle a_2|\hat{A}|a_2\rangle \end{bmatrix}, \\ T &= \begin{bmatrix} \langle a_1|\, b_1\rangle & \langle a_1|\, b_2\rangle \\ \langle a_2|\, b_1\rangle & \langle a_2|\, b_2\rangle \end{bmatrix}. \end{aligned}$$

one can write

$$H' = T^\dagger H T$$

where

$$\begin{aligned} H' &= \begin{bmatrix} \langle x_1|\hat{H}|x_1\rangle & \langle x_1|\hat{H}|x_2\rangle \\ \langle x_2|\hat{H}|b_1\rangle & \langle x_2|\hat{H}|x_2\rangle \end{bmatrix}, \\ H &= \begin{pmatrix} \langle 0|\hat{H}|0\rangle & \langle 0|\hat{H}|1\rangle \\ \langle 1|\hat{H}|0\rangle & \langle 1|\hat{H}|1\rangle \end{pmatrix} = \begin{pmatrix} \frac{\hbar\omega}{2} & 0 \\ 0 & \frac{3\hbar\omega}{2} \end{pmatrix}, \\ T &= \begin{bmatrix} \langle 0|\, x_1\rangle & \langle 0|\, x_2\rangle \\ \langle 1|\, x_1\rangle & \langle 1|\, x_2\rangle \end{bmatrix}. \end{aligned}$$

Noting that

$$\begin{aligned} T &= \begin{bmatrix} \langle 0|\, x_1\rangle & \langle 0|\, x_2\rangle \\ \langle 1|\, x_1\rangle & \langle 1|\, x_2\rangle \end{bmatrix} = T = \begin{bmatrix} \langle 0|\,\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & \langle 0|\,\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ \langle 1|\,\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & \langle 1|\,\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{bmatrix} \\ &\Rightarrow \; T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow T^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

we find

$$\begin{aligned} H' &= T^\dagger H T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} \frac{\hbar\omega}{2} & 0 \\ 0 & \frac{3\hbar\omega}{2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &\Rightarrow \; H' = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} \frac{\hbar\omega}{2} & \frac{\hbar\omega}{2} \\ \frac{3\hbar\omega}{2} & -\frac{3\hbar\omega}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\hbar\omega & -\hbar\omega \\ -\hbar\omega & 2\hbar\omega \end{pmatrix} \\ &\Rightarrow \; H' = \frac{\hbar\omega}{2}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

Equations Page

- *Some relations and constants*

$$\begin{aligned}
\left(\hat{A}\hat{B}\right)^{\dagger} &= \hat{B}^{\dagger}\hat{A}^{\dagger}, v = \frac{\lambda\omega}{2\pi}, v = \frac{c}{n}, k = \frac{2\pi}{\lambda} \Rightarrow k = \frac{\omega}{c}n, \varphi = \frac{\omega}{c}\left(n_y - n_x\right) l \\
\left|\pm X\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|+Z\right\rangle \pm \left|-Z\right\rangle\right], \\
\left|\pm Y\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|+Z\right\rangle \pm i\left|-Z\right\rangle\right] \\
\hbar &= h/2\pi = 1.055 \times 10^{-34} J.s = 6.582 \times 10^{-16} eV.s, c = 3.0 \times 10^{8} m/s
\end{aligned}$$

- *A set of vectors* $\{\left|a_1\right\rangle, \left|a_2\right\rangle, \left|a_3\right\rangle, ... \left|a_n\right\rangle\}$ *satisfying the condition*

$$\left\langle a_j \right| a_i\rangle = \delta_{ij}$$

are known as an orthonormal set of vectors. For an orthonormal complete set of vector, the completeness relation:

$$\sum_n \left|a_n\right\rangle\left\langle a_n\right| = 1.$$

- *Eigen value equation*

$$M\ \left|\vec{r}\right\rangle = \lambda\left|\vec{r}\right\rangle,$$

where λ *is the eigenvalue and* $\left|\vec{r}\right\rangle$ *is the eigenvector. The matrix representation*

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The eigenvalues are obtained from the condition

$$\begin{vmatrix} M_{11}-\lambda & M_{12} & M_{13} \\ M_{21} & M_{22}-\lambda & M_{23} \\ M_{31} & M_{32} & M_{33}-\lambda \end{vmatrix} = 0,$$

To find the eigenvectors we substitute the eigenvalues and solve the resulting equations.

- *For a state vector*

$$\left|\psi\right\rangle = \sum_{i=1}^{N} c_i\left|a_i\right\rangle$$

the average value (expectation value) and the standard deviation (uncertainty) for $\hat{A}$

$$\left\langle \hat{A}\right\rangle = \sum_{i=1}^{N}\left|c_i\right|^2 a_i = \left\langle\psi\right|\hat{A}\left|\psi\right\rangle, \left(\Delta\hat{A}\right) = \sqrt{\left\langle\hat{A}^2\right\rangle - \left\langle\hat{A}\right\rangle^2}.$$

- *The transformation equation from the basis* $\{\left|a_1\right\rangle, \left|a_2\right\rangle, ... \left|a_N\right\rangle\}$ *representation of the operator* $\hat{A}$ *to the basis* $\{\left|b_1\right\rangle, \left|b_2\right\rangle, ... \left|b_N\right\rangle\}$ *representation*

$$A' = T^{\dagger}AT$$

where

$$\begin{aligned}
A' &= \begin{bmatrix} \left\langle b_1\right|\hat{A}\left|b_1\right\rangle & \left\langle b_1\right|\hat{A}\left|b_2\right\rangle \\ \left\langle b_2\right|\hat{A}\left|b_1\right\rangle & \left\langle b_2\right|\hat{A}\left|b_2\right\rangle \end{bmatrix}, A = \begin{bmatrix} \left\langle a_1\right|\hat{A}\left|a_1\right\rangle & \left\langle a_1\right|\hat{A}\left|a_2\right\rangle \\ \left\langle a_2\right|\hat{A}\left|a_1\right\rangle & \left\langle a_2\right|\hat{A}\left|a_2\right\rangle \end{bmatrix}, \\
T &= \begin{bmatrix} \left\langle a_1\right| b_1\rangle & \left\langle a_1\right| b_2\rangle \\ \left\langle a_2\right| b_1\rangle & \left\langle a_2\right| b_2\rangle \end{bmatrix}.
\end{aligned}$$

- *Two none commuting operators and the uncertainty relation:*

$$\left[\hat{A}, \hat{B}\right] = i\hat{C} \Rightarrow \left(\Delta A\right)^2\left(\Delta B\right)^2 \geq \frac{1}{4}\left\langle\hat{C}\right\rangle^2.$$