Boundedness and asymptotic behavior of positive solutions for difference equations of exponential form

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Abstract
In this paper we study the boundedness and the asymptotic behavior of the positive solutions of the difference equation
\[ x_{n+1} = a + bx_n e^{-x_{n-1}}, \]
where \(a, b\) are positive constants, and the initial values \(x_{-1}, x_0\) are positive numbers. ©2015 All rights reserved.

Keywords: Difference equations, boundedness, asymptotic stability.


1. Introduction
In recent years, the behavior of positive solutions of the difference equations of exponential form has been one of the main topics in the theory of difference equations \([1, 2, 4, 5, 6]\). In particular, In \([2]\) the authors studied the existence of the equilibrium and the boundedness of solutions of the difference equation
\[ x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}, \]
where \(\alpha, \beta\) are positive constants and the initial values \(x_{-1}, x_0\) are positive numbers. Later in \([3]\), Fotiades studied the existence, uniqueness and attractivity of prime period two solution of this equation. For similar research on difference equation, we refer the reader to \([1, 4, 5, 6]\) and the references therein.

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Received 2015-5-4
In this paper, we investigate the global stability, the boundedness nature of the positive solutions of the difference equation

\[ x_{n+1} = a + bx_ne^{-x_{n-1}}, \]  

(1.1)

where the parameters \(a, b\) are positive numbers and the initial conditions \(x_{-1}, x_0\) are arbitrary nonnegative numbers. Equation (1.1) could be also viewed as a model in mathematical biology, in which case we consider \(a\) to be the immigration rate and \(b\) to be the population growth rate of one species \(x_n\).

2. Preliminaries

Let \(I\) be an interval of real numbers, and let \(f : I \times I \rightarrow I\) be a continuous function. Consider the difference equation

\[ x_{n+1} = f(x_n, x_{n-1}), n = 0, 1, \ldots, \]  

(2.1)

where the initial values \(x_{-1}, x_0 \in I\).

The main tool we will use is the following lemma which is a minor modification of Theorem 5.2 in [4].

**Lemma 2.1.** Support that \(f\) satisfies the following conditions:

(a) There exist positive numbers \(a, b\) with \(a < b\) such that \(a \leq f(x, y) \leq b\) for all \(x, y \in [a, b]\).

(b) \(f(x, y)\) is increasing in \(x \in [a, b]\) for each \(y \in [a, b]\), and \(f(x, y)\) is decreasing in \(y \in [a, b]\) for each \(x \in [a, b]\).

(c) (2.1) has no solutions of prime period two in \([a, b]\).

Then there exists exactly one equilibrium solution \(\bar{x}\) (2.1) which lies in \([a, b]\). Moreover, every solution of (2.1) with initial conditions \(x_{-1}, x_0 \in [a, b]\) converges to \(\bar{x}\).

Before we give the main result of this paper, we establish the existence and uniqueness of equilibrium of (1.1).

**Proposition 2.2.** Suppose that \(b < e^a\). Then (1.1) has a unique positive equilibrium \(\bar{x}\).

Here we omit its proof since it is similar as in [2].

3. Boundedness and the asymptotic behavior of Solutions

The following theorem gives a sufficient condition for every positive solution of (1.1) to be bounded.

**Theorem 3.1.** Every positive solution of (1.1) is bounded if

\[ b < e^a. \]  

(3.1)

**Proof.** Let \(\{x_n\}_{n=-1}^{\infty}\) be an arbitrary solution of Eq.(1.1). Observe that for all \(n \geq 2,

\[ x_{n+1} = a + bx_ne^{-x_{n-1}} \leq a + bx_ne^{-a}. \]  

(3.2)

We will now consider the non-homogeneous difference equations

\[ y_{n+1} = a + by_ne^{-a}, n = 2, 3, \ldots. \]  

(3.3)

From (3.3), an arbitrary solution \(\{y_n\}_{n=2}^{\infty}\) of (3.3) is given by

\[ y_n = r(b^{-a})^n + \frac{a}{1 - be^{-a}}, \]  

(3.4)
where \( r \) depends on the initial values \( y_2 \). Thus we see that relations (3.1) and (3.4) imply that \( y_n \) is a bounded sequence. Now we will consider the solution \( y_n \) of (3.3) such that

\[
y_2 = x_2. \tag{3.5}
\]

Thus from (3.2) and (3.5) we get

\[
x_n \leq y_n, \quad n \geq 2.
\]

Therefore it follows that \( x_n \) is bounded.

Next theorem will study the existence of invariant intervals of (1.1).

**Theorem 3.2.** Consider (1.1) where relations (3.1) hold. Then the following statements are true:

(i) The set

\[
[a, \frac{a}{1 - be - a}]
\]

is an invariant set for (1.1).

(ii) Let \( \varepsilon \) be an arbitrary positive number and \( x_n \) be an arbitrary solution of (1.1). We then consider the set

\[
I = [a, \frac{a + \varepsilon}{1 - be - a}]. \tag{3.6}
\]

Then there exists an \( n_0 \) such that for all \( n \geq n_0 \)

\[
x_n \in I. \tag{3.7}
\]

**Proof.** (i) Let \( x_n \) be a solution of (1.1) with initial values \( x_{-1}, x_0 \) such that

\[
x_{-1}, x_0 \in [a, \frac{a}{1 - be - a}]. \tag{3.8}
\]

Then from (1.1) and (3.8) we get

\[
a \leq x_1 = a + bx_0 e^{-x_{-1}} \leq a + b \frac{a}{1 - be - a} e^{-a} = \frac{a}{1 - be - a}.
\]

Then it follows by induction that

\[
a \leq x_n \leq \frac{a}{1 - be - a}, \quad n = 1, 2, \ldots
\]

This completes the proof of statement (i).

(ii) Let \( x_n \) be an arbitrary solution of (1.1). Therefore, from Theorem 3.1 we get

\[
0 < l = \liminf_{n \to \infty} x_n, \quad L = \limsup_{n \to \infty} x_n < \infty. \tag{3.9}
\]

It follows from (1.1) and (3.9) that

\[
L \leq a + bL e^{-l}, \quad l \geq a + bL e^{-L},
\]

which imply that

\[
a \leq L \leq \frac{a}{1 - be - a}.
\]

Thus from (1.1), we see that there exists an \( n_0 \) such that (3.7) holds true. This completes the proof of the proposition.

In the following, we will study the asymptotic behavior of the positive solutions of (1.1).
Theorem 3.3. Consider (1.1) where the initial values \( x_{-1}, x_0 \) are positive constants and \( a, b \) are positive constants satisfying
\[
b < e^a. \tag{3.10}
\]
Then (1.1) has a unique positive equilibrium \( \bar{x} \) such that
\[
\bar{x} \in [a, \frac{a}{1 - be^{-a}}]. \tag{3.11}
\]
Moreover, every positive solution of (1.1) tends to the unique positive equilibrium \( \bar{x} \) as \( n \to \infty \).

Proof. Following by Proposition 2.1 and Theorem 3.2, it suffices to show that any positive solution \( x_n \) converges to the unique positive equilibrium \( \bar{x} \) of (1.1). So, by lemma 2.1, we need to show that (1.1) has no positive solutions with prime period two.

Let \( x, y \in (a, \infty) \) be such that \( x = a + bye^{-x} \) and \( y = a + bxe^{-y} \). \tag{3.12}

It suffices to show that \( x = y, x > a, y > a \).

From (3.12) we get
\[
x = y - \frac{a}{be^{-y}}, \quad y = x - \frac{a}{be^{-x}}
\]
and so
\[
x = \frac{(x - a - abe^{-x})e^{\frac{x - a + abe^{-x}}{be^{-x}}}}{b^2}.
\]
Set
\[
F(x) = \frac{(x - a - abe^{-x})e^{\frac{x - a + abe^{-x}}{be^{-x}}}}{b^2} - x, \quad x \in (a, \infty).
\]
Since (1.1) has a unique positive equilibrium \( \bar{x} \), the following relation holds
\[
\bar{x} = \frac{a}{1 - be^{-\bar{x}}}. \tag{3.14}
\]
We claim that
\[
F'(\bar{x}) > 0. \tag{3.15}
\]
From (3.13) and (3.14), we have
\[
F'(\bar{x}) = e^{2\bar{x}}(\bar{x}^4 + (2 - 2a)\bar{x}^3 + (a^2 - 2a)\bar{x}^2 + 2a\bar{x} - a^2).
\]
To prove (3.15) is true, it suffices to prove that for \( u \geq a \),
\[
g(u) - h(u) > 0,
\]
\[
g(u) = u^4 + 2u^3 + a^2u^2 + 2au,
\]
\[
h(u) = 2au^3 + 2au^2 + a^2. \tag{3.17}
\]
From (3.17) we get
\[
g'(u) = 4u^3 + 6u^2 + 2a^2u + 2a, \quad h'(u) = 6au^2 + 4au,
\]
\[
g''(u) = 12u^2 + 12u + 2a^2, \quad h''(u) = 12au + 4a,
\]
\[
g'''(u) = 24u + 12, \quad h'''(u) = 12a,
\]
\[
g^{(4)}(u) = 24, \quad h^{(4)}(u) = 0. \tag{3.18}
\]
Now from (3.10) and (3.18), we have
\[ g^{(4)}(u) - h^{(4)}(u) = 24 > 0. \]
Since \( u \geq a > 0 \), we have
\[ g'''(u) - h'''(u) > g'''(a) - h'''(a) = 12a + 12 > 0. \] (3.19)
Using (3.19) and (3.10) we get
\[ g''(u) - h''(u) > g''(a) - h''(a) = 2a^2 + 8a > 0. \] (3.20)
Therefore from (3.20) and (3.10) it follows
\[ g'(u) - h'(u) > g'(a) - h'(a) = 2a^2 + 2a > 0. \] (3.21)
Hence from (3.21) and as \( u \geq a \), we get
\[ g(u) - h(u) > g(a) - h(a) = a^2 > 0, \]
which implies that (3.15) is true.
Since \( \bar{x} \) is a solution of the equation \( F(x) = 0 \) and (3.15) is satisfied there is \( \epsilon > 0 \) such that
\[ F(\bar{x} - \epsilon) < 0 \quad \text{and} \quad F(\bar{x} + \epsilon) > 0. \] (3.22)
Moreover from (3.13) we have
\[ F(a) = -\frac{a + ab}{b} < 0, \quad \lim_{x \to \infty} F(x) = +\infty. \] (3.23)
From (3.22) and (3.23) it follows that there exists exactly one root of \( F(x) = 0 \) in \((a, +\infty)\). So, (1.1) has no positive solutions of prime period two. The proof is complete. \( \Box \)

**Theorem 3.4.** Suppose that \( a \geq 2 \),
\[ b < \frac{2}{a + \sqrt{a^2 - 4a}} e^{1+a}. \] (3.24)
Then the equilibrium \( \bar{x} \) of (1.1) is locally asymptotically stable.

**Proof.** The linearized equation of (1.1) at the equilibrium \( \bar{x} \) is
\[ y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \cdots, \] (3.25)
where
\[ p = be^{-\bar{x}} = \frac{\bar{x} - a}{x} > 0, \quad q = -b\bar{x}e^{-\bar{x}} = a - \bar{x} < 0. \] (3.26)
From [2], a necessary and sufficient condition for absolute value of both eigenvalues of (3.25) less than one is
\[ |p| < 1 - q < 2. \] (3.27)
that is, by direct computation,
\[ b < \frac{2}{a + \sqrt{a^2 - 4a}} e^{1+a}, \] (3.28)
which imply \( \bar{x} \) is locally asymptotically stable. The proof is complete. \( \Box \)

**Remark 3.5.** If \( a \geq 2 \), \( b > \frac{2}{a + \sqrt{a^2 - 4a}} e^{1+a} \), the equilibrium \( \bar{x} \) is unstable.
Theorem 3.6. If \( a \geq 2 \),
\[
b < \min\{e^a, \frac{2}{a + \sqrt{a^2 - 4a}}e^{1+a}\},
\]
the equilibrium \( \bar{x} \) of (1.1) is globally asymptotically stable.

Proof. It follows immediately from Theorem 3.3 and Theorem 3.4.

Example 3.7. See Figure 1, (a) and (b) show the stability of equilibrium of (1.1) and (c) shows the unstable case whenever (3.24) is not satisfied.

(a) \( a = 2, b = 5 \).

(b) \( a = 4, b = 21 \).

(c) \( a = 2, b = 21 \).

Figure 1
Acknowledgements

This paper was funded by the National Natural Science Foundation of China (61363058), the Scientific Research Fund for Colleges and Universities of Gansu Province (2013B-007,2013A-016), Natural Science Foundation of Gansu Province (145RJZA232,145RJYA259) and Promotion Funds for Young Teachers in Northwest Normal University (NWNU-LKQN-12-14).

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