PHYS 4380 Quantum Mechanics I

Homework Assignment 05

Due date: October 9, 2018

Instructor: Dr. Daniel Erenso Name: _____

Mandatory problems: 1 & 5

Student signature:_____

Student Comment:_____
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Problem #	1	2	3	4	5	Score
Score	/	/	/	/	/	/100

1. Townsend 3.1 & 3.8

Townsend 3.1

Solution:

(a)

$$\begin{aligned}\left[\hat{A},\hat{B}+\hat{C}\right] &= \hat{A}\left(\hat{B}+\hat{C}\right)-\left(\hat{B}+\hat{C}\right)\hat{A}=\hat{A}\hat{B}+\hat{A}\hat{C}-\hat{B}\hat{A}-\hat{C}\hat{A}=\hat{A}\hat{B}-\hat{B}\hat{A}+\hat{A}\hat{C}-\hat{C}\hat{A}\\ &\Rightarrow \left[\hat{A},\hat{B}+\hat{C}\right]=\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right]\end{aligned}$$

(b)

$$\begin{aligned}\left[\hat{A},\hat{B}\hat{C}\right] &= \hat{A}\hat{B}\hat{C}-\hat{B}\hat{C}\hat{A}=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}=\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)\hat{C}+\hat{B}\left(\hat{A}\hat{C}-\hat{C}\hat{A}\right)\\ &\Rightarrow \left[\hat{A},\hat{B}\hat{C}\right]=\left[\hat{A},\hat{B}\right]\hat{C}+\hat{B}\left[\hat{A},\hat{C}\right]\end{aligned}$$

(c)

$$\begin{aligned}\left[\hat{A}\hat{B},\hat{C}\right] &= \hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B}=\hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}=\hat{A}\left(\hat{B}\hat{C}-\hat{C}\hat{B}\right)+\left(\hat{A}\hat{C}-\hat{C}\hat{A}\right)\hat{B}\\ &\Rightarrow \left[\hat{A}\hat{B},\hat{C}\right]=\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}\end{aligned}$$

Townsend 3.8

Solution: For an operator $\hat{C}$ defined by

$$\left[\hat{A},\hat{B}\right]=i\hat{C}\Rightarrow\hat{C}=-i\left[\hat{A},\hat{B}\right]$$

the Hermitian adjoint operator $\hat{C}^\dagger$ is given by

$$\begin{aligned}\hat{C}^\dagger &= \left\{-i\left[\hat{A},\hat{B}\right]\right\}^\dagger=(-i)^\dagger\left[\hat{A},\hat{B}\right]^\dagger=i\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)^\dagger=i\left(\left(\hat{A}\hat{B}\right)^\dagger-\left(\hat{B}\hat{A}\right)^\dagger\right)=i\left(\hat{B}^\dagger\hat{A}^\dagger-\hat{A}^\dagger\hat{B}^\dagger\right)\\ &= -i\left(\hat{A}^\dagger\hat{B}^\dagger-\hat{B}^\dagger\hat{A}^\dagger\right)\Rightarrow\hat{C}^\dagger=-i\left[\hat{A}^\dagger,\hat{B}^\dagger\right]\end{aligned}$$

We can find

$$\hat{C}^\dagger=-i\left[\hat{A},\hat{B}\right]=\hat{C}$$

if and only if both $\hat{A}$ and $\hat{B}$ are Hermitian,

$$\hat{A}^\dagger=\hat{A},\hat{B}^\dagger=\hat{B}$$

2. Townsend 3.9 and 3.10

Solution: Townsend 3.9

Recalling that the eigenvalue equation for $\hat{S}_x$ and $\hat{S}_y$ for a spin-half particle is

$$\hat{S}_x\left|\pm X\right\rangle=\pm\frac{\hbar}{2}\left|\pm X\right\rangle,\hat{S}_y\left|\pm Y\right\rangle=\pm\frac{\hbar}{2}\left|\pm Y\right\rangle$$

for an eigenstate

$$\left|\psi\right\rangle=\left|\pm Z\right\rangle$$

which can be expressed in terms of the x and y basis

$$\begin{aligned}\left|+X\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|+Z\right\rangle+\left|-Z\right\rangle\right),\left|-X\right\rangle=\frac{1}{\sqrt{2}}\left(\left|+Z\right\rangle-\left|-Z\right\rangle\right)\Rightarrow\left|\pm Z\right\rangle=\frac{1}{\sqrt{2}}\left(\left|+X\right\rangle\pm\left|-X\right\rangle\right)\\ &\Rightarrow \left|\psi\right\rangle=\left|\pm Z\right\rangle=\frac{1}{\sqrt{2}}\left(\left|+X\right\rangle\pm\left|-X\right\rangle\right)\end{aligned}$$

and

$$
\begin{aligned}
|+Y\rangle &= \frac{1}{\sqrt{2}}\left(|+Z\rangle + i\,|-Z\rangle\right), |-Y\rangle = \frac{1}{\sqrt{2}}\left(|+Z\rangle - i\,|-Z\rangle\right) \\
&\Rightarrow \quad |+Z\rangle = \frac{1}{\sqrt{2}}\left(|-Y\rangle + |+Y\rangle\right), |-Z\rangle = \frac{i}{\sqrt{2}}\left(|-Y\rangle - |+Y\rangle\right) \\
&\Rightarrow \quad |\psi\rangle = |+Z\rangle = \frac{1}{\sqrt{2}}\left(|-Y\rangle + |+Y\rangle\right) \text{ or } |\psi\rangle = |-Z\rangle = \frac{i}{\sqrt{2}}\left(|-Y\rangle - |+Y\rangle\right)
\end{aligned}
$$

we have

$$
\left\langle \hat{S}_x \right\rangle = \langle\psi|\,\hat{S}_x\,|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle +X| \pm \langle -X|\right)\hat{S}_z\frac{1}{\sqrt{2}}\left(|+X\rangle \pm |-X\rangle\right)
$$

$$
= \frac{1}{2}\left(\langle +X| \pm \langle -X|\right)\left[\frac{\hbar}{2}\,|+X\rangle \pm \left(-\frac{\hbar}{2}\right)|-X\rangle\right] \Rightarrow \left\langle \hat{S}_x \right\rangle = \frac{1}{2}\left(\frac{\hbar}{2}\,\langle +X|\,+X\rangle + \left(-\frac{\hbar}{2}\right)\langle -X|\,+X\rangle\right) = 0
$$

$$
\left\langle \hat{S}_x^2 \right\rangle = \langle\psi|\,\hat{S}_x^2\,|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle +X| \pm \langle -X|\right)\hat{S}_z^2\frac{1}{\sqrt{2}}\left(|+X\rangle \pm |-X\rangle\right)
$$

$$
= \frac{1}{2}\left(\langle +X| \pm \langle -X|\right)\left[\left(\frac{\hbar}{2}\right)^2|+X\rangle \pm \left(-\frac{\hbar}{2}\right)^2|-X\rangle\right] = \frac{1}{2}\left[\left(\frac{\hbar}{2}\right)^2\langle +X|\,+X\rangle + \left(-\frac{\hbar}{2}\right)^2\langle -X|\,+X\rangle\right]
$$

$$
\Rightarrow \left\langle \hat{S}_x^2 \right\rangle = \frac{\hbar^2}{4}
$$

$$
\left\langle \hat{S}_y \right\rangle = \langle\psi|\,\hat{S}_y\,|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle +Y| \pm \langle -Y|\right)\hat{S}_y\frac{1}{\sqrt{2}}\left(|+Y\rangle \pm |-Y\rangle\right)
$$

$$
= \frac{1}{2}\left(\langle +Y| \pm \langle -Y|\right)\left[\frac{\hbar}{2}\,|+Y\rangle \pm \left(-\frac{\hbar}{2}\right)|-Y\rangle\right] \Rightarrow \left\langle \hat{S}_y \right\rangle = 0
$$

$$
\left\langle \hat{S}_y^2 \right\rangle = \langle\psi|\,\hat{S}_y^2\,|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle +Y| \pm \langle -Y|\right)\hat{S}_y^2\frac{1}{\sqrt{2}}\left(|+Y\rangle \pm |-Y\rangle\right)
$$

$$
= \frac{1}{2}\left(\langle +Y| \pm \langle -Y|\right)\left[\left(\frac{\hbar}{2}\right)^2|+Y\rangle \pm \left(-\frac{\hbar}{2}\right)^2|-Y\rangle\right] \Rightarrow \left\langle \hat{S}_y^2 \right\rangle = \frac{\hbar^2}{4}
$$

Then the uncertainties becomes

$$
\Delta S_y = \sqrt{\left\langle \hat{S}_y^2 \right\rangle - \left\langle \hat{S}_y \right\rangle^2} = \frac{\hbar}{2}, \Delta S_x = \sqrt{\left\langle \hat{S}_x^2 \right\rangle - \left\langle \hat{S}_x \right\rangle^2} = \frac{\hbar}{2} \Rightarrow \Delta S_y \Delta S_x = \frac{\hbar^2}{4}
$$

Noting that for

$$
|\psi\rangle = |\pm Z\rangle
$$

we have

$$
\left\langle \hat{S}_z \right\rangle = \langle\psi|\,\hat{S}_z\,|\psi\rangle = \pm\frac{\hbar}{2}
$$

so that

$$
\frac{\hbar\left|\left\langle \hat{S}_z \right\rangle\right|}{2} = \frac{\hbar^2}{4}
$$

Therefore, what we found is

$$
\Delta S_y \Delta S_x = \frac{\hbar\left|\left\langle \hat{S}_z \right\rangle\right|}{2}
$$

3. (a) Show that

$$\langle\psi|\,\hat{A}\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)|\psi\rangle=\langle\psi|\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2|\psi\rangle\,.$$

(b) Show that the quadrature operators

$$\hat{a}_1=\frac{\hat{O}^\dagger+\hat{O}}{2},\hat{a}_2=\frac{i\left(\hat{O}^\dagger-\hat{O}\right)}{2}, \tag{1}$$

are Hermitian and any operator $\hat{O}$.

Solution: (a)

$$\begin{aligned}\langle\psi|\,\hat{A}\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)|\psi\rangle&=\langle\psi|\left(\hat{A}-\left\langle\hat{A}\right\rangle+\left\langle\hat{A}\right\rangle\right)\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)|\psi\rangle=\langle\psi|\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2|\psi\rangle+\langle\psi|\left\langle\hat{A}\right\rangle\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)|\psi\rangle\\&=\langle\psi|\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2|\psi\rangle+\left\langle\hat{A}\right\rangle\langle\psi|\,\hat{A}\,|\psi\rangle-\langle\psi|\left\langle\hat{A}\right\rangle^2|\psi\rangle=\langle\psi|\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2|\psi\rangle+\left\langle\hat{A}\right\rangle^2-\left\langle\hat{A}\right\rangle^2\\&\Rightarrow\langle\psi|\,\hat{A}\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)|\psi\rangle=\langle\psi|\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2|\psi\rangle\,,\end{aligned} \tag{2}$$

(b)

$$\hat{a}_1^\dagger=\left(\frac{\hat{O}^\dagger+\hat{O}}{2}\right)^\dagger=\frac{\left(\hat{O}^\dagger\right)^\dagger+\hat{O}^\dagger}{2}=\frac{\hat{O}+\hat{O}^\dagger}{2}=\hat{a}_1, \tag{3}$$

$$\hat{a}_2^\dagger=\left(\frac{i\left(\hat{O}^\dagger-\hat{O}\right)}{2}\right)^\dagger=-i\left(\frac{\left(\hat{O}^\dagger\right)^\dagger-\hat{O}^\dagger}{2}\right)=-i\left(\frac{\hat{O}-\hat{O}^\dagger}{2}\right)=\frac{i\left(\hat{O}^\dagger-\hat{O}\right)}{2}=\hat{a}_2, \tag{4}$$

4. Suppose we rotated the vector $\vec{A}=(A_x,A_y,A_z)$ by an angle φ about the y-axis and found a new vector $\vec{A}'=\left(A'_x,A'_y,A'_z\right)$. The projection of the vector $\vec{A}$ on the x-z plane makes an angle θ from the positive z-axis (try to make 3D vectors visualization like the one in Fig.**??** in my note). .Show that the rotation matrix is given by

$$R(\varphi j)=\begin{bmatrix}\cos(\varphi)&0&\sin(\varphi)\\0&1&0\\-\sin(\varphi)&0&\cos(\varphi)\end{bmatrix}. \tag{5}$$

5. Following the same approach we followed in class show that

$$\left[\hat{J}_z,\hat{J}_x\right]=i\hbar\hat{J}_y$$

Solution: Using

$$R(\Delta\varphi i)=\begin{bmatrix}1&0&0\\0&1-\frac{(\Delta\varphi)^2}{2}&-\Delta\varphi\\0&\Delta\varphi&1-\frac{(\Delta\varphi)^2}{2}\end{bmatrix},R(\Delta\varphi k)=\begin{bmatrix}1-\frac{(\Delta\varphi)^2}{2}&-\Delta\varphi&0\\\Delta\varphi&1-\frac{(\Delta\varphi)^2}{2}&0\\0&0&1\end{bmatrix}. \tag{6}$$

we may write

$$\begin{aligned}&R(\Delta\varphi k)\,R(\Delta\varphi i)\\&=\begin{bmatrix}1-\frac{(\Delta\varphi)^2}{2}&-\Delta\varphi&0\\\Delta\varphi&1-\frac{(\Delta\varphi)^2}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1-\frac{(\Delta\varphi)^2}{2}&-\Delta\varphi\\0&\Delta\varphi&1-\frac{(\Delta\varphi)^2}{2}\end{bmatrix}\\&=\begin{bmatrix}\left(1-\frac{(\Delta\varphi)^2}{2}\right)^2&-\left(1-\frac{(\Delta\varphi)^2}{2}\right)\Delta\varphi&(\Delta\varphi)^2\\\Delta\varphi&\left(1-\frac{(\Delta\varphi)^2}{2}\right)^2&-\Delta\varphi\left(1-\frac{(\Delta\varphi)^2}{2}\right)\\0&\Delta\varphi&1-\frac{(\Delta\varphi)^2}{2}\end{bmatrix}.\end{aligned} \tag{7}$$

and for the reverse order rotation

$$
\begin{aligned}
& R(\Delta\varphi i)\, R(\Delta\varphi k) \\
= & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{(\Delta\varphi)^2}{2} & -\Delta\varphi \\ 0 & \Delta\varphi & 1-\frac{(\Delta\varphi)^2}{2} \end{bmatrix} \begin{bmatrix} 1-\frac{(\Delta\varphi)^2}{2} & -\Delta\varphi & 0 \\ \Delta\varphi & 1-\frac{(\Delta\varphi)^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= & \begin{bmatrix} \left(1-\frac{(\Delta\varphi)^2}{2}\right)^2 & -\Delta\varphi & 0 \\ \Delta\varphi\left(1-\frac{(\Delta\varphi)^2}{2}\right) & \left(1-\frac{(\Delta\varphi)^2}{2}\right)^2 & -\Delta\varphi \\ (\Delta\varphi)^2 & \Delta\varphi\left(1-\frac{(\Delta\varphi)^2}{2}\right) & 1-\frac{(\Delta\varphi)^2}{2} \end{bmatrix}. \qquad (8)
\end{aligned}
$$

There follows that

$$
\begin{aligned}
& R(\Delta\varphi k)\, R(\Delta\varphi i) - R(\Delta\varphi i)\, R(\Delta\varphi k) \\
= & \begin{bmatrix} \left(1-\frac{(\Delta\varphi)^2}{2}\right)^2 & -\left(1-\frac{(\Delta\varphi)^2}{2}\right)\Delta\varphi & (\Delta\varphi)^2 \\ \Delta\varphi & \left(1-\frac{(\Delta\varphi)^2}{2}\right)^2 & -\Delta\varphi\left(1-\frac{(\Delta\varphi)^2}{2}\right) \\ 0 & \Delta\varphi & 1-\frac{(\Delta\varphi)^2}{2} \end{bmatrix} \\
& - \begin{bmatrix} \left(1-\frac{(\Delta\varphi)^2}{2}\right)^2 & -\Delta\varphi & 0 \\ \Delta\varphi\left(1-\frac{(\Delta\varphi)^2}{2}\right) & \left(1-\frac{(\Delta\varphi)^2}{2}\right)^2 & -\Delta\varphi \\ (\Delta\varphi)^2 & \Delta\varphi\left(1-\frac{(\Delta\varphi)^2}{2}\right) & 1-\frac{(\Delta\varphi)^2}{2} \end{bmatrix} \\
= & \begin{bmatrix} 0 & \frac{(\Delta\varphi)^3}{2} & (\Delta\varphi)^2 \\ \frac{(\Delta\varphi)^3}{2} & 0 & \frac{(\Delta\varphi)^3}{2} \\ -(\Delta\varphi)^2 & \frac{(\Delta\varphi)^3}{2} & 0 \end{bmatrix} \qquad (9)
\end{aligned}
$$

Since we can make our rotation as small as we want to, we can set $(\Delta\varphi)^3 \simeq 0$. This leads to

$$
R(\Delta\varphi k)\, R(\Delta\varphi i) - R(\Delta\varphi i)\, R(\Delta\varphi k) = \begin{bmatrix} 0 & 0 & (\Delta\varphi)^2 \\ 0 & 0 & 0 \\ -(\Delta\varphi)^2 & 0 & 0 \end{bmatrix}
$$

Suppose if one make a rotation by an angle of $\Delta\varphi' = (\Delta\varphi)^2$, in a counterclockwise direction about the y axis, the rotation matrix can be written as

$$
\begin{aligned}
R\left((\Delta\varphi)^2 j\right) &= \begin{bmatrix} \cos\left((\Delta\varphi)^2\right) & 0 & \sin\left((\Delta\varphi)^2\right) \\ 0 & 1 & 0 \\ -\sin\left((\Delta\varphi)^2\right) & 0 & \cos\left((\Delta\varphi)^2\right) \end{bmatrix} \simeq \begin{bmatrix} 1-\frac{(\Delta\varphi)^4}{2} & 0 & (\Delta\varphi)^2 \\ 0 & 1 & 0 \\ -(\Delta\varphi)^2 & 0 & 1-\frac{(\Delta\varphi)^4}{2} \end{bmatrix} \qquad (10) \\
&\simeq \begin{bmatrix} 1 & 0 & (\Delta\varphi)^2 \\ 0 & 1 & 0 \\ -(\Delta\varphi)^2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & (\Delta\varphi)^2 \\ 0 & 0 & 0 \\ -(\Delta\varphi)^2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (11) \\
&\Rightarrow \begin{bmatrix} 0 & 0 & (\Delta\varphi)^2 \\ 0 & 0 & 0 \\ -(\Delta\varphi)^2 & 0 & 0 \end{bmatrix} = R\left((\Delta\varphi)^2 j\right) - I \qquad (12)
\end{aligned}
$$

where

$$
I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

is the identity matrix. One can then rewrite

$$R\left(\Delta\varphi k\right)R\left(\Delta\varphi i\right)-R\left(\Delta\varphi i\right)R\left(\Delta\varphi k\right)=\left[\begin{array}{ccc}0 & 0 & \left(\Delta\varphi\right)^{2}\\ 0 & 0 & 0\\ -\left(\Delta\varphi\right)^{2} & 0 & 0\end{array}\right]=R\left(\left(\Delta\varphi\right)^{2}j\right)-I$$

and it can be put in terms of the corresponding operators in the form

$$\hat{R}\left(\Delta\varphi k\right)\hat{R}\left(\Delta\varphi i\right)-\hat{R}\left(\Delta\varphi i\right)\hat{R}\left(\Delta\varphi k\right)=\hat{R}\left(\left(\Delta\varphi\right)^{2}j\right)-\hat{I}. \tag{13}$$

We recall that these operators (for spin-one particles like a photon) in terms of the rotation generators can be expressed as

$$\begin{aligned}\hat{R}\left(\Delta\varphi i\right) &= e^{-i\frac{\hat{J}_x}{\hbar}\Delta\varphi}=\hat{I}-i\frac{\hat{J}_x}{\hbar}\Delta\varphi+\frac{1}{2}\left(\frac{\hat{J}_x}{\hbar}\Delta\varphi\right)^{2}...,\\ \hat{R}\left(\Delta\varphi k\right) &= e^{-i\frac{\hat{J}_z}{\hbar}\Delta\varphi}=\hat{I}-i\frac{\hat{J}_z}{\hbar}\Delta\varphi+\frac{1}{2}\left(\frac{\hat{J}_z}{\hbar}\Delta\varphi\right)^{2}...,\\ \hat{R}\left(\left(\Delta\varphi\right)^{2}j\right) &= e^{-i\frac{\hat{J}_y}{\hbar}\left(\Delta\varphi\right)^{2}}=\hat{I}-i\frac{\hat{J}_y}{\hbar}\left(\Delta\varphi\right)^{2}...\Rightarrow\hat{R}\left(\left(\Delta\varphi\right)^{2}j\right)-\hat{I}=-i\frac{\hat{J}_y}{\hbar}\left(\Delta\varphi\right)^{2}...\end{aligned} \tag{14}$$

so that the commutation relation becomes

$$\begin{aligned}&\left[\hat{I}-i\frac{\hat{J}_z}{\hbar}\Delta\varphi+\frac{1}{2}\left(\frac{\hat{J}_z}{\hbar}\Delta\varphi\right)^{2}...\right]\left[\hat{I}-i\frac{\hat{J}_x}{\hbar}\Delta\varphi+\frac{1}{2}\left(\frac{\hat{J}_x}{\hbar}\Delta\varphi\right)^{2}...\right]\\ &-\left[\hat{I}-i\frac{\hat{J}_x}{\hbar}\Delta\varphi+\frac{1}{2}\left(\frac{\hat{J}_x}{\hbar}\Delta\varphi\right)^{2}...\right]\left[\hat{I}-i\frac{\hat{J}_z}{\hbar}\Delta\varphi+\frac{1}{2}\left(\frac{\hat{J}_z}{\hbar}\Delta\varphi\right)^{2}...\right]\\ &=-i\frac{\hat{J}_y}{\hbar}\left(\Delta\varphi\right)^{2}....\end{aligned} \tag{15}$$

The zero and first order in $\Delta\varphi$ cancel out and keeping only up to the second order in $\Delta\varphi$ on the left side of this equation, one can easily find

$$\begin{aligned}&\left[-i\frac{\hat{J}_z}{\hbar}\Delta\varphi\right]\left[-i\frac{\hat{J}_x}{\hbar}\Delta\varphi\right]-\left[-i\frac{\hat{J}_x}{\hbar}\Delta\varphi\right]\left[-i\frac{\hat{J}_z}{\hbar}\Delta\varphi\right]\\ &=-i\frac{\hat{J}_y}{\hbar}\left(\Delta\varphi\right)^{2}\Rightarrow\hat{J}_z\hat{J}_x-\hat{J}_x\hat{J}_z=i\hbar\hat{J}_y\Rightarrow\left[\hat{J}_z,\hat{J}_x\right]=i\hbar\hat{J}_y.\end{aligned}$$