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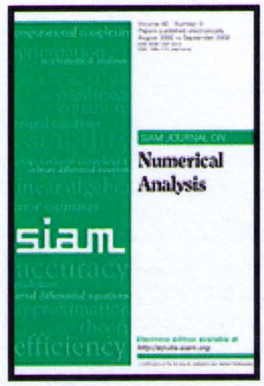

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SIAM J. Numer. Anal.
Vol. 34, No. 4, pp. 1472–1482, August 1997

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SWAPPING EDGES OF ARBITRARY TRIANGULATIONS TO ACHIEVE THE OPTIMAL ORDER OF APPROXIMATION*

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Abstract. In the representation of scattered data by smooth *pp* (:= piecewise polynomial) functions, perhaps the most important problem is to find an optimal triangulation of the given sample sites (called vertices). Of course, the notion of optimality depends on the desirable properties in the approximation or modeling problems. In this paper, we are concerned with optimal approximation order with respect to the given order r of smoothness and degree k of the polynomial pieces of the smooth *pp* functions. We will only consider C^1 *pp* approximation with $r = 1$ and $k = 4$. The main result in this paper is an efficient method for triangulating any finitely many arbitrarily scattered sample sites, such that these sample sites are the only vertices of the triangulation, and that for any discrete data given at these sample sites, there is a C^1 piecewise quartic polynomial on this triangulation that interpolates the given data with the fifth order of approximation.

Key words. approximation orders, bivariate splines, edge swapping, optimal triangulation, scattered data representation

AMS subject classifications. Primary, 41A25, 41A63; Secondary, 41A05, 41A15, 65D07

PII. S0036142994273537

1. Introduction.

Among the many methods for scattered data interpolation (approximation) on a polygonal domain Ω (see [5]), a class of methods is based on triangulating Ω with the sample sites as the only vertices of the triangulation. A collection $\Delta = \{\tau_i\}_{i=1}^N$ of triangles τ_i in $\mathbb{R}^2$ is called a triangulation of a finite set V of sample sites v_i, if (i) the vertices of the triangles are precisely the sample sites v_i, (ii) the union $\Omega := \bigcup_{i=1}^N \tau_i$ is a connected set, and (iii) the intersection of any two adjacent triangles in Δ is either a common vertex or a common edge. We will call V the vertex set of the triangulation Δ. In the study of *pp* (:= piecewise polynomial) functions on a triangulation Δ, the notation $S_k^r(\Delta)$ is used to denote the subspace of $C^r(\Omega)$ of all *pp* functions with total degree $\leq k$ and with grid lines given by the edges of Δ.

In general, a vertex set V has many different triangulations. One of the most important problems in the representation of scattered data defined on V by C^r *pp* functions is to find an "optimal triangulation" of V, and by this we mean that (i) the set V of sample sites is the same as the set of vertices of the triangulation, and (ii) the space of *pp* functions with degree k and smoothness order r on this triangulation achieves the highest order of approximation. This order, of course, cannot exceed $k+1$. We only study C^1 piecewise quartic polynomial approximation, with $r = 1$ and $k = 4$. For any finitely many arbitrarily scattered sample sites in $\mathbb{R}^2$, we have an efficient algorithm for constructing a triangulation with these sample sites as the only vertices such that a C^1 piecewise quartic polynomial interpolation scheme on this triangulation can be formulated to interpolate any given data on these sample sites

*Received by the editors August 26, 1994; accepted for publication (in revised form) November 7, 1995. This research was partially supported by NSF grants DMS 92-06928 and DMS 95-05460.
http://www.siam.org/journals/sinum/34-4/27353.html

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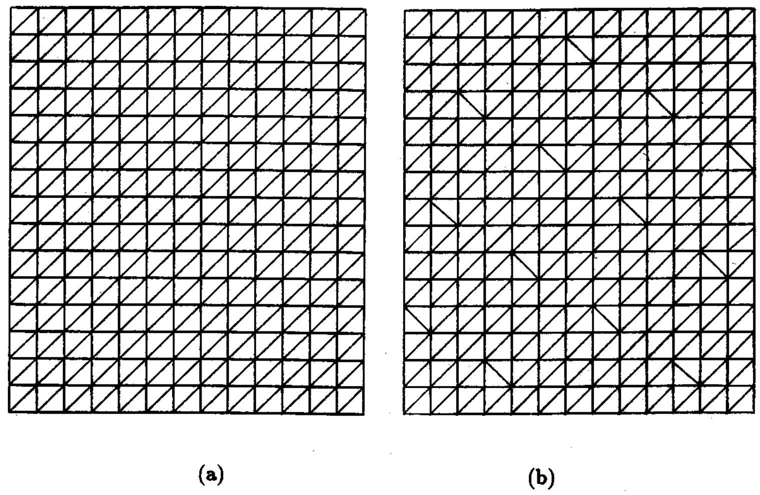

FIG. 1. *Triangulations* $\Delta^{(1)}$ *and* $\widehat{\Delta}$ *of a finite rectangular set of uniformly spaced lattice points.*

with the highest (or fifth) order of approximation. To demonstrate the nontriviality of this problem, let us consider a finite rectangular block of uniformly spaced lattice points. It is well known that with the three-directional mesh $\Delta^{(1)}$ (as shown in Fig. 1 (a)), the spline space $S_4^1(\Delta^{(1)})$ only admits the fourth order of approximation (cf. [2]), while we will show in this paper that the new triangulation $\widehat{\Delta}$ (as shown in Fig. 1 (b)), obtained by applying our algorithm, guarantees the fifth order of approximation from $S_4^1(\widehat{\Delta})$.

This paper is organized as follows. In the next section, we give a brief review of the B-net representation of *pp* functions and introduce the notion of type-O triangulations, where "O" stands for optimal order of approximation. In section 3, we construct a locally supported basis for the space $S_4^1(\Delta)$ over any type-O triangulation Δ. In section 4, we give an interpolation scheme from the space $S_4^1(\Delta)$ over any type-O triangulation Δ that provides the optimal (or fifth) order of approximation. Based on edge-swapping, an efficient algorithm for constructing a type-O triangulation Δ of an arbitrary finite vertex set, so that the fifth order of approximation from $S_4^1(\Delta)$ is achieved, will be discussed in the final section of this paper.

2. Preliminaries and statement of main results. The B-net representations of *pp* functions will play an important role in our discussion. For completeness, we give a very brief review of such representations. More details can be found in the spline literature, such as [3].

Recall that to any positive integer k, a Bernstein–Bèzier polynomial basis of degree k is given by

$$B_{\alpha,\tau}(x) = \begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix} \xi^\alpha, \quad \alpha = (\alpha_u, \alpha_v, \alpha_w) \in \mathbb{Z}_+^3, \quad |\alpha| := \alpha_u + \alpha_v + \alpha_w = k,$$

where $\xi = (\xi_u, \xi_v, \xi_w)$ is the barycentric coordinate of x with respect to a given triangle $\tau = [u, v, w]$ and

$$\xi^\alpha = \xi_u^{\alpha_u} \xi_v^{\alpha_v} \xi_w^{\alpha_w} \quad \text{and} \quad \begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix} = \frac{|\alpha|!}{\alpha_0!\alpha_1!\alpha_2!}.$$

The points

$$x_{\alpha,\tau} = \frac{1}{k}(\alpha_0 u + \alpha_1 v + \alpha_2 w), \qquad |\alpha| = k,$$

are usually called *domain points* of the triangle τ, and the set of all domain points on Δ will be denoted by X. For each function $s \in S_k^0(\Delta)$, write

$$s(x) = \sum_{|\alpha|=k} b_{\alpha,\tau} B_{\alpha,\tau}(x), \qquad \alpha \in \mathbb{Z}_+^3\ ,\ x \in \tau \in \Delta.$$

Then the map

$$b_s \in \mathbb{R}^X : \quad x_{\alpha,\tau} \mapsto b_{\alpha,\tau}, \quad \alpha \in \mathbb{Z}_+^3, \quad |\alpha| = k, \quad \tau \in \Delta\ ,$$

is called the B-net representation of s. We have the following estimates (cf. [4]).

LEMMA 1. *The B-net representation $b_s \in \mathbb{R}^X$ of any $s \in S_k^0(\Delta)$ satisfies*

$$||s||_\infty \le ||b_s||_\infty \le C_k ||s||_\infty\ .$$

Now, let $\tau = [u, v, w]$ and $\tilde{\tau} = [u, v, \tilde{w}]$ be two triangles in Δ with common edge $e = [u, v]$. Also let $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ denote the unit coordinate vectors in $\mathbb{R}^3$. Then it is well known that the C^1 smoothness conditions across the edge e for any $s \in S_4^1(\Delta)$, with B-net coordinates $b_{\alpha,\tau}$, is determined by the relation

$$b_{\alpha+\mathbf{e}^3,\tilde{\tau}} = c_1 b_{\alpha+\mathbf{e}^1,\tau} + c_2 b_{\alpha+\mathbf{e}^2,\tau} + c_3 b_{\alpha+\mathbf{e}^3,\tau}\ , \tag{1}$$

where $\alpha = (\alpha_u, \alpha_v, 0) \in \mathbb{Z}_+^3$ with $\alpha_u + \alpha_v = 3$, and c_i, $i = 1, 2, 3$, are the barycentric coordinates of $\tilde{w}$ with respect to τ; i.e.,

$$c_1 = \frac{area[\tilde{w}, v, w]}{area[u, v, w]},\ c_2 = \frac{area[u, \tilde{w}, w]}{area[u, v, w]},\ c_3 = \frac{area[u, v, \tilde{w}]}{area[u, v, w]}\ . \tag{2}$$

In what follows, we denote by V_I, for a given triangulation Δ with vertex set V, the set of all interior vertices in Δ, and call $V_b := V \setminus V_I$ the set of all boundary vertices in Δ. We also denote the collection of all edges in Δ by E and the collection of all interior edges in Δ by E_I.

Recall that the degree of any vertex $v \in V$, which we will denote by $\deg(v)$, is the number of edges emanating from v. If $\deg(v)$ is an even integer, then we say that v is an even-degree vertex; otherwise, v is called an odd-degree vertex. In addition, an interior vertex v is called a singular vertex if (i) its degree is 4 and (ii) it is the intersection of two straight line segments. If e_{j-1}, e_j, e_{j+1} are three consecutive edges with a common vertex v, then the edge e_j is called degenerate with respect to v, provided that the two edges e_{j-1} and e_{j+1} are colinear.

To introduce the notion of a type-O triangulation, we need to classify its vertices. A vertex u will be called a *type-O vertex* of a triangulation Δ, if u satisfies at least one of the following.

(a) u is a boundary vertex of Δ.

(b) $u \in V_I$ with $\deg(u) = 4$.

(c) $u \in V_I$ and $\deg(u)$ is an odd integer.

(d) $u \in V_I$ and there exists a vertex v of Δ that satisfies either (i) $v \in V_I$ and $\deg(v) = 4$ or $\deg(v) =$ an odd integer, or (ii) $v \in V_b$, such that $[u, v]$ is a nondegenerate edge of Δ with respect to u.

DEFINITION. *A triangulation of V with only type-*O *vertices is called a* type-O triangulation.

The reason for introducing the notion of type-O triangulations is the following.

THEOREM 1. *Any type-O triangulation Δ admits the fifth order of approximation from the space $S_4^1(\Delta)$.*

This theorem is a consequence of Theorem 4, to be established in section 4.

The second main result of this paper is that any arbitrary finite vertex set, with the exception of a one-dimensional set (i.e., those whose vertices lie on a straight line), admits a type-O triangulation. In fact, in section 5, we will give an algorithm for changing any triangulation to a type-O triangulation simply by edge-swapping, so that no new vertices are introduced.

3. A local basis over any type-O triangulation. A subset $\mathcal{P}$ of domain points will be called a *determining set* of the space $S_k^r(\Delta)$, if the zero function is the only function in $S_k^r(\Delta)$ whose B-net representation vanishes on $\mathcal{P}$. Such a determining set $\mathcal{P}$ is called a *minimal determining set* if there is no determining set with fewer elements. Clearly, $\mathcal{P}$ is a determining set for $S_k^r(\Delta)$ if and only if the linear map $s \mapsto b_s|_{\mathcal{P}}$, defined on $S_k^r(\Delta)$, is one-one; also, $\mathcal{P}$ is a minimal determining set for $S_k^r(\Delta)$ if and only if this one-one linear map is also onto. To construct a local basis of the space $S_4^1(\Delta)$ for a type-O triangulation Δ, we choose a minimal determining set $\mathcal{P}$ for $S_4^1(\Delta)$ so that the B-net coordinates $b(x)$, $x \in X \setminus \mathcal{P}$, are dependent only upon a very small subset of the B-net coordinates close to the x. This has some important practical advantages. For example, a local perturbation of a data set will only alter the interpolant locally, and a locally supported basis derived from such a determining set will ensure that the space $S_4^1(\Delta)$ has the fifth order of approximation.

A minimal determining set $\mathcal{P}$ that ensures these properties for a type-O triangulation Δ is constructed as follows.

Step 1. To any even-degree interior vertex $u \in V_I$ with $\deg(u) \geq 6$, there is a nondegenerate edge $[u, v]$ with respect to u, such that v satisfies (a), (b), or (c). We put a check-mark on such an edge $[u, v]$ (the mark is shown as a diamond in the example in Fig. 2), and assign the midpoints of all unmarked edges to $\mathcal{P}$.

Step 2. Every vertex u of the triangulation Δ will be assigned a certain triangle τ_u, among all the triangles with u as the common vertex. If u has $\deg(u) = 4$ but is nonsingular, we label its four neighbor vertices $u_1, \dots, u_4$ in the counterclockwise direction in such a way that both sets $\{u, u_1, u_3\}$ and $\{u_2, u_3, u_4\}$ are noncolinear, and choose $\tau_u = [u, u_1, u_2]$. Otherwise, any triangle attached to u may be assigned to u. Note that τ_u, which has been assigned to u, may be assigned to another neighbor vertex again.

Step 3. If u has $\deg(u) = 4$ but is nonsingular with $\tau_u = [u, v, w]$, say, then we add the points u and $(3u + v)/4$ to $\mathcal{P}$. For any other vertex u with $\tau_u = [u, v, w]$, we add the points $(3u + v)/4$ and $(3u + w)/4$ as well as u itself to $\mathcal{P}$.

Step 4. To any vertex u which is not an interior vertex with odd degree, we add the point $(2u + v + w)/4$ to $\mathcal{P}$, where again $\tau_u = [u, v, w]$ is the triangle assigned to u.

In addition to the set $\mathcal{P}$ as constructed above, we need the following notation.

Let u be any vertex and $\tau = [u, v, w]$ be any triangle with u as one of its vertices. We consider the set

$$X_{u,\tau}^n = \{x_{\alpha,\tau} : \ \alpha_u = k - n\}$$

of domain points on $\tau \in \Delta$ associated with u, and call the sets

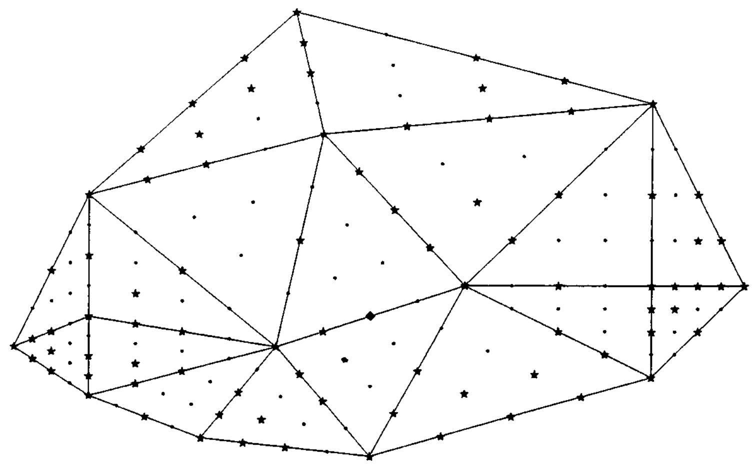

FIG. 2. *Elements ($\star$) in the determining set.*

$$R_u^n = \bigcup_{\tau \ni u} X_{u,\tau}^n = \{x_{\alpha,\tau} : \quad \tau \in \Delta, \quad \alpha_u = k - n\}$$

and

$$D_u^n = \bigcup_{j=0}^{n} R_u^j = \{x_{\alpha,\tau} : \quad \alpha_u \geq k - n, \quad \tau \in \Delta\}$$

the nth *ring* and nth *disk* around u, respectively (cf. [1]).

The following result implies that $\mathcal{P}$ is a minimal determining set for the space $S_4^1(\Delta)$.

THEOREM 2. *For each $b : \mathcal{P} \mapsto \mathbb{R}$, there exists a unique $g \in S_4^1(\Delta)$ such that the B-net representation b_g of g satisfies*

$$b_g|_{\mathcal{P}} = b \quad and \quad \|b_g\|_\infty \leq \|b\|_\infty \ ,$$

where C is a positive constant independent of b and the mesh size $|\Delta| := \sup_{\tau\in\Delta} \operatorname{diam} \tau$.

Proof. We first prove that $\mathcal{P}$ is a determining set. Let us arrange the vertices in an appropriate order and extend b to the B-net coordinates b_g of g as follows.

(i) If u is an even-degree interior vertex with $\deg(u) \geq 6$, then according to Lemma 6 in [1], we can determine the b_g values on all of the domain points in D_u^2 by using the given values on $\mathcal{P}$.

(ii) Any of the remaining vertices must be a boundary vertex, an odd-degree vertex, or an interior vertex with $\deg(u) = 4$.

From (i) and the choice of the set $\mathcal{P}$, we see that the midpoints of all the edges are uniquely determined. Therefore, by Lemmas 2–5 in [1], it is clear that the b_g values can be uniquely determined on all the domain points in D_u^2.

Next, it is easy to see that there are $(3 + \deg(u))$ points in $\mathcal{P} \cap D_u^2$ for any nonsingular interior vertex u, and $(4 + \deg(v))$ points in $\mathcal{P} \cap D_v^2$ for any singular or boundary vertex v. Therefore, we have

$$|\mathcal{P}| = 3|V_I| + 4|V_b| + |E| + \sigma,$$

where, as usual, $|A|$ denotes the cardinality of a set A. Furthermore, it follows from [9] that

$$\dim S_4^1(\Delta) \geq 3|V_I| + 4|V_b| + |E| + \sigma = |\mathcal{P}| \ .$$

Hence $\mathcal{P}$ must also be a minimal determining set for $S_4^1(\Delta)$, and consequently, the dimension of the space $S_4^1(\Delta)$ over a type-O triangulation Δ is given by

$$\dim S_4^1(\Delta) = 3|V_I| + 4|V_b| + |E| + \sigma = |\mathcal{P}| \ ,$$

so that the extension b_g of b is also unique.

Furthermore, we see that b_g satisfies the C^1 smoothness conditions on D_u^2. For the vertices discussed in (i), note that $b_g|_{D_u^2\setminus\mathcal{P}}$ can be determined by using the explicit smoothness conditions in (1). It follows from (2) that the coefficients c_i are bounded by a constant independent of b and $|\Delta|$. This is also valid for any vertex with the exception of the nonsingular ones with degree 4. If u is a nonsingular vertex with $\deg(u) = 4$, then by Lemma 5 in [1], the determinant of the coefficient matrix for solving $b_g|_{D_u^2\setminus\mathcal{P}}$ is a nonzero according to our choice of $\mathcal{P}$, and it is also independent of b and $|\Delta|$. This completes the proof of the theorem. □

Let

$$d := |\mathcal{P}| = \dim S_4^1(\Delta)$$

and write

$$\mathcal{P} = \{x_1, \dots, x_d\} \subset X \ .$$

Also, let $\{b_1, \dots, b_d\} \subset \mathbb{R}^X$ be the "dual" of $\mathcal{P}$, defined as follows: (i) $b_i(x_j) = \delta_{ij},\ i,j = 1, \dots, d$, and (ii) for each $x \in X \setminus \mathcal{P}$, $b_i(x)$ is uniquely determined by the smoothness condition (1) by following the procedure as described in the proof of Theorem 2. Let us now consider the function $s_i \in S_4^1(\Delta)$ with B-net representation b_i, $i = 1, \dots, d$. Then it is clear that $\{s_1, \dots, s_d\}$ is a basis of $S_4^1(\Delta)$. We will show that this basis is local.

For this purpose, let us first recall the notion of a closed star $\overline{St}(u)$ with vertex u [7, p. 135] in a triangulation Δ defined as the union of all the triangles in Δ with u as the common vertex. More precisely, let us call $\overline{St}(u)$ a 1-star of u and denote it by $\overline{St}^1(u)$. For $m \geq 1$, the m-star $\overline{St}^m(u)$ of u can be defined inductively as the union of all the triangles in Δ which have at least one common vertex with a certain $(m-1)$-star $\overline{St}^{m-1}(u)$. Analogous to the definition of vertex splines, a spline is called an *m-star spline* if its support is no larger than $\overline{St}^m(u)$ for some vertex $u \in \Delta$. Hence, a 1-star spline is a vertex spline. We have the following result.

THEOREM 3. *For a type-O triangulation Δ, the basis $\{s_i\}_{i=1}^d$ of $S_4^1(\Delta)$ defined above is a locally supported basis with*

$$\mathrm{supp}(s_i) \subset \overline{St}^3(v_i)$$

for some vertices $v_i \in V,\ i = 1, \cdots, d$.

Proof. Following the procedure as described in the proof of Theorem 2, the b_i values of s_i on D_u^2 for even-degree interior vertices u with $\deg(u) \geq 6$ are determined prior to the b_i values on D_v^2 for any other vertex v. We divide our discussion into three cases.

(i) For $x_i \in \mathcal{P} \cap D_u^1$, since $b_i = 0$ outside of $\overline{St}(u)$, it is clear that $\mathrm{supp}(s_i) \subset \overline{St}(u)$. Now we may assume that $x_i \in \mathcal{P} \cap R_u^2$.

(ii) Suppose that $x_i \in D_u^2$ and u is one of the odd-degree vertices, boundary vertices, or even-degree interior vertices with $\deg(u) = 4$. If x_i is not the midpoint of an edge, then $\mathrm{supp}(s_i) \subset \overline{St}(u)$. On the other hand, for a midpoint of some edge $[u, v]$, if v is an even-degree interior vertex with $\deg(v) \geq 6$, and the edge $[v, w]$, with some vertex $w \neq u$, is check-marked as in Step 1 in the choice of $\mathcal{P}$ for vertex v (instead of u), then

$$\mathrm{supp}(s_i) \subset \overline{St}(u) \bigcup \overline{St}(v) \bigcup \overline{St}(w) \subset \overline{St}^2(v) \ .$$

Otherwise,

$$\mathrm{supp}(s_i) \subset \overline{St}(u) \bigcup \overline{St}(v) \subset \overline{St}^2(u) \ .$$

(iii) Finally, if u does not satisfy (ii), then u is an even-degree interior vertex with $\deg(u) \geq 6$. If x_i is not a midpoint of an edge, then $\mathrm{supp}(s_i) \subset \overline{St}(u)$. On the other hand, for a midpoint x_i of some edge $[u, u']$, suppose that u' is not an even-degree interior vertex with $\deg(u') \geq 6$. Then there is an edge $[u, v]$ which is check-marked in Step 1 in the choice of $\mathcal{P}$ for the vertex u. Hence,

$$\mathrm{supp}(s_i) \subset \overline{St}(u) \bigcup \overline{St}(v) \bigcup \overline{St}(u') \subset \overline{St}^2(u) \ .$$

Otherwise, by the choice of the determining set in $\mathcal{P}$, there are edges $[u, v]$ and $[u', v']$, check-marked as in Step 1 for the vertices u and u', respectively, such that v and v' are among the type-O vertices that satisfy (a), (b), or (c), and

$$\mathrm{supp}(s_i) \subset \overline{St}(u) \bigcup \overline{St}(u') \bigcup \overline{St}(v) \bigcup \overline{St}(v') \subset \overline{St}^3(u) \ .$$

In summary, for any $x_i \in \mathcal{P}$, the corresponding basis function $s_i \in S_4^1(\Delta)$ has support $\mathrm{supp}(s_i) \subset \overline{St}^3(u_i)$ for some vertex $u_i \in V$. This completes the proof of the theorem. □

4. An interpolation scheme and its approximation power. Since the minimal determining set $\mathcal{P}$ contains the vertex set V, any discrete data set with V as the set of sample sites admits an interpolant from $S_4^1(\Delta)$. In this section, we construct an explicit interpolation scheme for $S_4^1(\Delta)$, where Δ is a type-O triangulation, and apply this scheme to prove that the space $S_4^1(\Delta)$ achieves the fifth order of approximation.

Let (u, z_u), $u \in V$, be any discrete data set. For convenience, let us assume that this data set is derived from some continuous function f on $\Omega = \bigcup_{\tau \in \Delta} \tau$, i.e.,

$$f(u) = z_u \ , \qquad u \in V \ .$$

We will first construct a *pp* function $g \in S_4^0(\Delta)$ such that

$$g(u) = f(u) \ , \qquad u \in V \ .$$

In the course of our construction, we will show that our g is actually in C^1, so that $g \in S_4^1(\Delta)$ and is uniquely determined. Since our method is linear, this procedure induces a linear operator T from $C(\Omega)$ to $S_4^1(\Delta)$ defined by

$$T : \, f \mapsto g \ , \qquad f \in C(\Omega) \ . \tag{3}$$

Interpolation scheme.

Step 1. Let s be the *pp* function in $S_4^0(\Delta)$ which interpolates f at all the domain points on Δ. Then it is clear that

$$s(u) = f(u) \ , \qquad u \in V \ .$$

Furthermore, let us consider the B-net representation

$$s = \sum_{|\alpha|=k} b_s(x_{\alpha,\tau}) B_{\alpha,\tau}$$

of s, and set

$$b_g(x) = b_s(x) \ , \qquad x \in \mathcal{P} \ .$$

Step 2. Determine the remaining b_g values on $X \setminus \mathcal{P}$ by using the smoothness conditions in (1) in the order as described in the proof of Theorem 2. Let g be the *pp* function with B-net representation b_g. Then $g \in S_4^1(\Delta)$.

For a triangle $\tau \in \Delta$ with vertices u, v, and w, we consider the neighborhood

$$\Omega(\tau) = \overline{St}^2(u) \bigcup \overline{St}^2(v) \bigcup \overline{St}^2(w) \tag{4}$$

of τ. Then we have the following.

LEMMA 2. *The linear operator T defined in* (3) *satisfies*

(i) $Tp = p$ *for any polynomial* $p \in \pi_4$;

(ii) $\|Tf|_\tau\|_\infty \le C\|f|_{\Omega(\tau)}\|_\infty$, *where C is a positive constant independent of f.*

Proof. The first part of the lemma is obvious from the definition of the operator T. That the supports of the basis functions s_i of $S_4^1(\Delta)$ satisfy

$$\text{supp}(s_i) \subset \Omega(\tau), \quad \text{for some} \ \ \tau \in \Delta \ ,$$

follows from the proof of Theorem 3. Let $g(x) = (Tf)(x) := \sum_i c_i s_i(x)$, $x \in \tau$, $\tau \in \Delta$. According to Theorem 3, we have $s_i(x) \neq 0$ only if the corresponding domain point x_i lies in $\Omega(\tau)$. Therefore, the number of nonzero values of the c_i's is bounded from above by some positive constant C. Moreover, $\|b_{s_i}\| \le C \max_{x \in \mathcal{P} \cap \Omega(\tau)} |b_{s_i(x)}| = C$. Hence, by Lemma 1 and the definition of s_i, we have $\|s_i\| \le \|b_{s_i}\| \le C$ and $\max_i |c_i| \le \|b_g\|_\infty = \max_{x_i \in \Omega(\tau) \cap \mathcal{P}} |b_g|$. It follows that

$$|Tf(x)| \le C \max_{x \in \Omega(\tau) \cap \mathcal{P}} |b_g(x)| \le C\|f|_{\Omega(\tau)}\|_\infty \ , \qquad x \in \tau \in \Delta \ .$$

This completes the proof of the lemma. □

We are now in a position to prove the following result.

THEOREM 4. *Let Δ be a type-O triangulation. Then the linear (approximation) operator T defined in* (3) *has the fifth order of approximation; i.e.,*

$$\|Tf - f\| \le C\|f^{(5)}\| \ |\Delta|^5 \ , \qquad f \in C^5(\Delta) \ .$$

Consequently,

$$\text{dist}_\infty(f, S_4^1(\Delta)) \le C\|f^{(5)}\| \ |\Delta|^5 \ , \qquad f \in C^5(\Delta) \ ,$$

where C is a positive constant independent of f and $|\Delta|$.

Proof. Fix any $\tau \in \Delta$ and $x \in \tau$. Let $f \in C^5(\Omega)$. Then there is a polynomial $p \in \pi_4$ that interpolates f at point x, i.e.,

$$p(x) = f(x) , \tag{5}$$

and satisfies

$$|f(y) - p(y)| \leq C\|f^{(5)}\| \, |\Delta|^5 \tag{6}$$

for any $y \in \Omega(\tau)$ as defined by (4), where C is an absolute constant. By applying (5), Lemma 2, and (6) consecutively, it follows that

$$|f(x) - (Tf)(x)| = |T(f-p)(x)| \leq C\|(f-p)|_{\Omega(\tau)}\| \leq C\|f^{(5)}\| \, |\Delta|^5 .$$

Since this inequality holds for any $x \in \Delta$, we have

$$\|Tf - f\| \leq C\|f^{(5)}\| \, |\Delta|^5 .$$

This completes the proof of the theorem. ∎

It is easy to see that any odd-degree triangulation and the four-direction mesh are type-O triangulations. Therefore, we have the following.

COROLLARY 1. (a) *If a triangulation Δ consists only of odd-degree interior vertices, then there is an interpolation scheme from $S_4^1(\Delta)$ that yields the fifth order of approximation.*

(b) *For the four-direction mesh $\Delta^{(2)}$, the space $S_4^1(\Delta^{(2)})$ has the fifth order of approximation, and an interpolation scheme can be used to achieve this optimal approximation order.*

5. Construction of type-O triangulations. The main result of this paper will be established in this section, namely: to an arbitrary finite set V of sample sites, there is an optimal triangulation $\widehat{\Delta}$, with the sample sites as its only vertices. Optimality means that $S_4^1(\widehat{\Delta})$ has the fifth order of approximation. Our main idea is first to start with any triangulation Δ with the given points in V as its only vertices, and then change Δ to a type-O triangulation $\widehat{\Delta}$ by an edge-swapping process.

Every interior edge e of a triangulation Δ is the diagonal of a quadrilateral Q_e, which is the union of two triangles of Δ with common edge e. Following [8], we say that e is a swappable edge if Q_e is convex and no three of its vertices are colinear. If an edge e of a triangulation Δ is swappable, then we can create a new triangulation by swapping the edge. That is, if $v_1, \ldots, v_4$ are the vertices of Q_e ordered in the counterclockwise direction, and if e has endpoints v_1 and v_3, then the swapped edge has endpoints v_2 and v_4. Two vertices in Δ will be called *neighbors* of each other if they are the endpoints of the same edge in Δ. Hence, while v_1 and v_3 are neighbors in the original triangulation Δ, v_2 and v_4 become neighbors in the new triangulation after the edge e is swapped.

For any given set of sample sites, it is clear that with the exception of those that are colinear, there is a triangulation with these sample sites as its only vertices. Let Δ be a triangulation associated with the given set V, and let V_O be the set of all type-O vertices in Δ. Set

$$\tilde{V} = V \setminus V_O .$$

If $u \in \tilde{V}$, then u and all its neighbors with nondegenerate edges with respect to u must be even-degree vertices with $\deg(u) \geq 6$. We claim that, for every interior vertex

u with $n := \deg(u) \geq 5$, there is a swappable edge $e \in E_u$, where E_u denotes the set of all edges with common vertex $u \in V$. Indeed, if the neighbors u_i, $i = 1, \ldots, n$, of u are ordered in the counterclockwise direction, then

$$\overline{S}t^1(u) = \bigcup_{i=1}^{n} [u_i, u, u_{i+1}] \ ,$$

where $u_{n+1} := u_1$, and by setting $\alpha_i = \angle u_{i-1} u_i u_{i+1}$, we have

$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi \ .$$

Therefore, at least three of the α_i's are smaller than π. Let $\theta_i := \angle u_i u u_{i+1}$, $i = 1, \ldots, n$. Then, since $n \geq 5$, at most two of the values $\theta_i + \theta_{i+1}$, $i = 1, \ldots, n-1$, are greater than or equal to π. Hence, there is at least one vertex u_i such that both $\angle u_{i-1} u u_{i+1}$ and $\angle u_{i-1} u_i u_{i+1}$ are less than π. Therefore, the quadrilateral $Q := [u_{i-1}, u_i, u_{i+1}, u]$ is convex, and hence, the edge $[v, v_i]$ is swappable.

Now we are ready to describe our *Swapping Algorithm* for constructing a type-O triangulation $\widehat{\Delta}$, starting with any triangulation Δ.

Swapping Algorithm.

Do while ($\tilde{V} \neq \emptyset$)
Pick any vertex u in $\tilde{V}$ and consider its neighbors.
Pick any neighbor v of u so that the edge $[u, v]$ is swappable.
Swap $[u, v]$, yielding a new edge $[u', v']$.
Form a subset of $\tilde{V}$ by deleting from $\tilde{V}$ all the neighbors w of $w' := u$, v, u', or v', with $[w, w']$ being a nondegenerate edge with respect to w.
Call this subset $\tilde{V}$.
Enddo

The new triangulation obtained by applying this Swapping Algorithm is denoted by $\widehat{\Delta}$. Let $\widehat{V} := V$. We use $\widehat{V}_I$ and $\widehat{V}_b$ to denote the sets of interior and boundary vertices of $\widehat{\Delta}$, respectively, and define $\widehat{E}$, $\widehat{E}_I$, and $\widehat{E}_b$ in a similar way.

In the following, we give a rough estimate of the number of swapping steps required to obtain $\widehat{\Delta}$ from a given Δ. In the Swapping Algorithm, we only swap the edge with at least one even-degree interior vertex u with $\deg(u) \geq 6$, and once an edge has been swapped, there are at least two even-degree interior vertices u with $\deg(u) \geq 6$ that are changed to odd-degree vertices. Thus, every time we swap an edge, at least two even-degree interior vertices with $\deg(u) \geq 6$ do not have to be considered at the later stages of the Swapping Algorithm. Therefore, the number of steps required to perform an edge-swapping in the Swapping Algorithm is bounded from above by $\lfloor L/2 \rfloor$, where L is the number of even-degree interior vertices with $\deg(u) \geq 6$ in Δ.

It is clear that the triangulations Δ and $\widehat{\Delta}$ have the same number of triangles, singular vertices, interior and boundary vertices, and edges. Hence, it follows that

$$\dim\, S_4^1(\widehat{\Delta}) = \dim\, S_4^1(\Delta) \ .$$

Furthermore, it is clear that $\widehat{\Delta}$ is a type-O triangulation (since $\tilde{V} = \emptyset$). Therefore, as a consequence of Theorem 4 in section 4, we have the following.

THEOREM 5. *Every finite set V of sample sites admits a triangulation $\widehat{\Delta}$, such that $S_4^1(\widehat{\Delta})$ has the fifth order of approximation.*

We call $\widehat{\Delta}$ an optimal triangulation of the finite set V. In comparison with the refinement algorithms, our method has some advantages. First, instead of working on a given triangulation, as is usually done, we construct a triangulation with the sample sites as the only vertices and achieve the optimal order of approximation. This allows us to avoid subdividing the triangles to introduce new vertices (where data values are not available). Second, our Swapping Algorithm is efficient and does not change the dimension of the spline space over the original triangulation.

Acknowledgments. The authors are very grateful to the referees for their valuable comments and suggestions, which helped to improve the writing of this paper.

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