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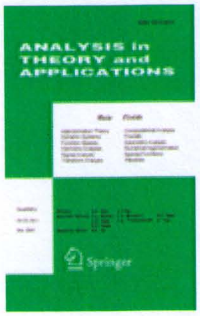

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SPACES OF BIVARIATE SPLINE FUNCTIONS OVER TRIANGULATION *

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Absract

We consider the spaces of bivariate C^{μ}−splines of degree k defined over arbitrary triangulations of a polygonal domain. We get an explicit formula for the dimension of such spaces when $k \geqslant 3\mu+2$ and construct a local basis for them. The dimension formula is valid for any polygonal domain even it is complex connected, and the formula is sharp since it evaluates the lower−bound which was given by Schumaker in [11].

§ 1 Introduction

As usual, let Ω be a subets of R^2, and let $\triangle = \{\tau_i\}_1^N$ be a collection of closed triangles such that

i) For all i,j, if $i \neq j$, the intersection $\tau_i \cap \tau_j$ is either empty, their common dege or their common vertex.

ii) $\Omega = \bigcup_1^N \tau_i$.

Then we call $\triangle$ a triangulation of Ω.

Given a positive integer k,we denote by $\prod_k$ the space of all polynomials in two variables with total degree $\leqslant k$. For a triangulation $\triangle$ of Ω, let

$$S_{k,\triangle} = \{s;\ \ s|_{\tau_i} \in \prod_k, \quad i = 1,\cdots,N\}$$

be the linear space of splines defined over $\triangle$ and

$$S^{\mu}_{k,\triangle} = S_{k,\triangle} \cap C^{\mu}. \tag{1.1}$$

We call $S^{\mu}_{k,\triangle}$ the spaces of bivariate polynomial splines of degree $\leqslant k$ and smoothness μ associated with the partition $\triangle$.

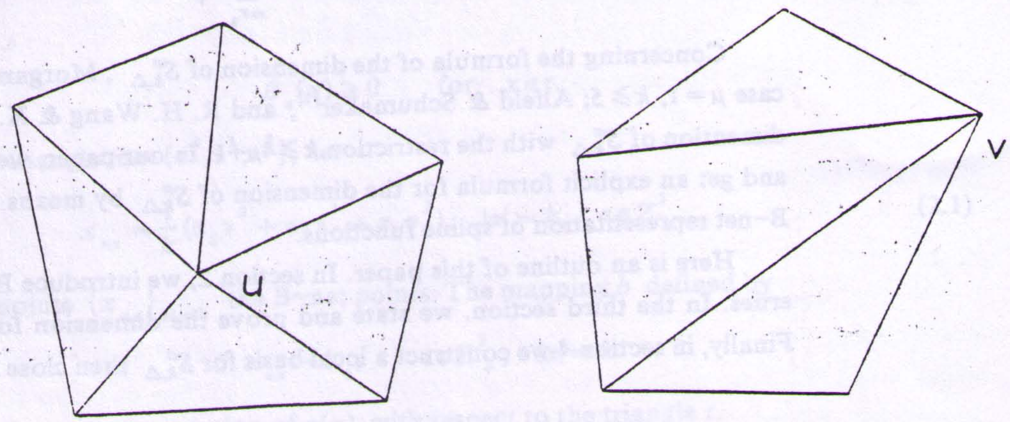
* The author wishes to thank sincerely professor Rongqing Jia for his encouragement and advice

It is clear that $S = S^{\mu}_{k,\triangle}$ is a linear space. Our main results of this paper is about computing its dimension and constructing a local basis. The work in this regard was initiated Strang[12], Morgan & Scott [10], and Schumaker [11]. Here we follow them and introduce some notation first.

Given a triangulation $\triangle$ of Ω, we denote by E, E_0, V and V_0 the sets of edges, interior edges, vertices and interior vertices of $\triangle$ respectively. Let

$$E_b = E \backslash E_0, \quad \text{and} \quad V_b = V \backslash V_0 \tag{1.2}$$

Given a vertex $v \in V$, we use e_v to denote the number of edges with different slopes attached to v. The cardinal number of a set A is denoted by $|A|$.



a) A standard cell with an interior vertex v b) A standard cell with respect to v

Fig.1 Standard cells

A set Ω triangulated by $\triangle = \{\tau_i\}$ is called a standard cell with respect to v, if all triangles of $\triangle$ have the vertex v in common. Figure 1 shows two typical examples of standard cell. The follwoing results which is about dimension of bivariate splines spaces defined over standard cells was proved by Schumaker [11]

Theorem S₁ *Let Ω be a standared cell triangulated by $\triangle$ and v the interior vertex. Then*

$$\dim(S^{\mu}_{k,\triangle}) = \alpha + \beta|E_0| - \gamma + \sigma_v, \tag{1.3}$$

where

$$\alpha = \frac{(k+1)(k+2)}{2}, \quad \beta = \frac{(k-\mu)(k-\mu+1)}{2},$$

$$\gamma = \frac{(k+1)(k+2)}{2} - \frac{(\mu+1)(\mu+2)}{2},$$

$$\sigma_v = \sum_{j=1}^{k-\mu} (\mu + j + 1 - je_v)_+ \tag{1.4}$$

In general cases of partition $\triangle$, Schumaker gave out a lower−bound for the dimension of $S^{\mu}_{k,\triangle}$ in the same paper. But it is valid only for simply connected domaims (see section 3 of out paper).

Theorem S_2 *Let $\triangle$ be a triangulation of Ω. Then*

$$\dim(S^{\mu}_{k,\triangle}) \geqslant \alpha + \beta|E_0| - \gamma|V_0| + \sigma, \tag{1.5}$$

where α,β,γ and σ_v are given by (1.4), *while*

$$\sigma = \sum_{v\in V_0} \sigma_v. \tag{1.6}$$

Concerning the formula of the dimension of $S^{\mu}_{k,\triangle}$, Morgan & Scott [10] considered the case $\mu=1$, $k\geqslant 5$; Alfeld & Schumaker[1], and R. H. Wang & X. G. Lu respectively got the dimention of $S^{\mu}_{k,\triangle}$ with the restriction $k\geqslant 4\mu+1$. In our paper, we consider the case $k\geqslant 3\mu+2$ and get an explicit formula for the dimension of $S^{\mu}_{k,\triangle}$ by means of an important method—B−net representation of spline functions.

Here is an outline of this paper. In section 2, we introduce B−net and its relative properties. In the third section, we state and prove the dimension formula of the space $S^{\mu}_{k,\triangle}$. Finally, in section 4 we construct a local basis for $S^{\mu}_{k,\triangle}$ then close the paper with remarks.

§ 2 B−net

In this section we briefly review B−net representation of splines and the relative properties. We use the standard multi−index notation. For $\alpha = (\alpha_0,\alpha_1,\alpha_2)\in Z^3_+$, the length $|\alpha|$ of α is defined by $|\alpha| = \alpha_0+\alpha_1+\alpha_2$. Moreover,

$$\alpha! = \alpha_0!\,\alpha_1!\,\alpha_2!, \quad \binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}.$$

Let $\tau = \{v^0, v^1, v^2\}$ denote the triangle with vertices v^0, v^1 and v^2. If τ is not degenerated, then any $x\in R^2$ can be uniquely expressed as

$$x = \xi_0 v^0 + \xi_1 v^1 + \xi_2 v^2, \quad \xi_0 + \xi_1 + \xi_2 = 1.$$

The 3−tuple $\xi = (\xi_0,\xi_1,\xi_2)$ is called the barycentric coordinate of x with respect to the triangle τ.

For any $a\in Z^3_+$, we denote

$$B_\alpha(x) := \binom{|\alpha|}{\alpha}\xi^\alpha,$$

where $\xi^{\alpha}=\xi_0^{\alpha_0}\xi_1^{\alpha_1}\xi_2^{\alpha_2}$.

It is evident that $\{B_\alpha;\ |\alpha|=k\}$ froms a basis for Π_k, hence any $p\in\Pi_k$ can be expressed as

$$p(x)=\sum_{|\alpha|=k} b_\alpha B_\alpha(x).$$

The coefficients $\{b_\alpha\}_{|\alpha|=k}$ are called the Bézier coordinates of $p(x)$ with respect to the triangle τ.

Clearly, we have

$$\sum_{|\alpha|=k} B_\alpha(x)=1$$

and

$$B_\alpha(x)\geqslant 0 \quad \text{for} \quad x\in\tau.$$

For the triangle $\tau=\{v^0,v^1,v^2\}$, let

$$x_{\alpha,\tau}=\frac{1}{k}(\alpha_0 v^0+\alpha_1 v^1+\alpha_2 v^2),\quad |\alpha|=k,\quad \alpha\in Z^3_+. \tag{2.1}$$

We call the points $\{x_{\alpha,\tau}\}_{|\alpha|=k}$ the B–net points. The mapping b defined by

$$b: x_{\alpha,\tau}\rightarrow b_\alpha,\quad \alpha\in Z^3_+,\quad |\alpha|=k$$

is called the B–net representation of $p(x)$ with respect to the triangle τ.

Associated with the triangulation $\triangle$, the space $S^{\mu}_{k,\triangle}$ of bivariate splines on $\triangle$ is defined by (1.1). Particularly, $S^0_{k,\triangle}$ is the space of continuous spline functions. Now we consider the B–net representation of a bivariate spline in the space $S^{\mu}_{k,\triangle}$. Let

$$P=P_k(\triangle):=\bigcup_{\tau\in\triangle}\{x_{\alpha,\tau};\ |\alpha|=k\}.$$

where $x_{\alpha,\tau}$ is given by (2.1).

For $s\in S^0_{k,\triangle}$ is agreement with a polynomial $p_i\in\Pi_k$ on each $\tau_i\in\triangle$, we have

$$p_i(x)=\sum_{|\alpha|=k} b_{\alpha,\tau_i} B_{\alpha,\tau_i}.$$

Therefore, a mapping b_s on P can be defined by

$$b_s:\ x_{\alpha,\tau_i}\rightarrow b_{\alpha,\tau_i},\quad |\alpha|=k,\quad \tau_i\in\triangle. \tag{2.2}$$

The mapping b_s is called the B–net representation of the spline function s. Clearly, b_s is well defined provided s is continuous. In this way, we obtain a one–to –one correspondence between $s\in S^0_{k,\triangle}$ and its B–net representation b_s. So, for any $s\in S^0_{k,\triangle}$,we have the following expression

$$s(x) = \sum_{|\alpha|=k} b_s(x_{\alpha,\tau}) B_{\alpha,\tau}(x), \qquad x \in \tau, \quad \tau \in \triangle.$$

As usual, the space of all real functions defined on P will be denoted by R^P. Then $S^0_{k,\triangle}$ is isomorphic to R^P. Therefore, we have

$$\dim(S^0_{k,\triangle}) = \dim(R^P) = |P|. \tag{2.3}$$

We are going to introduce the smoothness conditions of splines in terms of B−net representation. First of all, we consider the case that two triangles have only one vertex in common. We need the following lemma (see[4],[8])

Lemma 2.1. *Let* $z = \zeta_0 v^0 + \zeta_1 v^1 + \zeta_2 v^2$, $\zeta_0 + \zeta_1 + \zeta_2 = 0$. *For* $f \in S^\mu_{k,\triangle}$, b_f *is the B−net representation of f. Then*

$$(D_z)^r f(x) = \sum_{|\alpha|=k-r} \left(\sum_{|\beta|=r} \binom{r}{\beta} b_f(x_{\alpha+\beta}) \right) B_{\alpha,\tau}(x)$$

for all $0 \leq r \leq \mu$, *where* $\alpha, \beta \in Z^3_+$, *and* $D_z f$ *denotes the derivative of f in the direction of the vector z.*

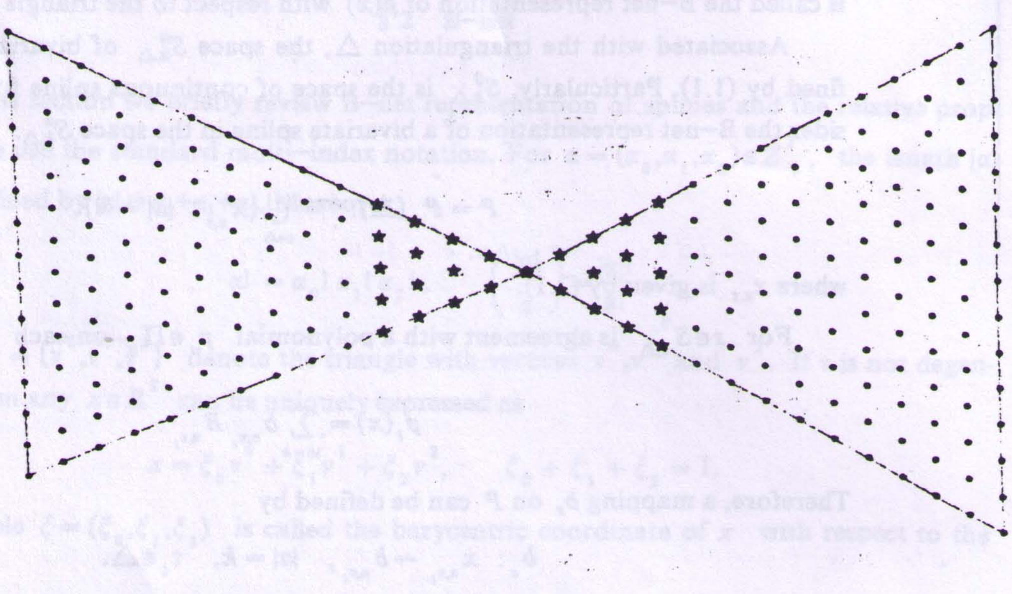


Fig.2 The points (*) in the set $P(v,\mu)$

(for $k = 14, \mu = 4$)

Assume that $\tau=\{v,u,w\}$ and $\tau'=\{v,u',w'\}$ are two triangles with exactly one vertex v in common (see Figure 2), and that $x_{\alpha,\tau}=\frac{\alpha_0}{k}v+\frac{\alpha_1}{k}u+\frac{\alpha_2}{k}w$ and $x_{\alpha,\tau'}=\frac{\alpha_0}{k}v+\frac{\alpha_1}{k}u'+\frac{\alpha_2}{k}w'$ are B–net points in τ and τ' respectively. Let $P(v,\mu)$ be the collection of those B–net points in $\tau\bigcup\tau'$ whose barycentric coordinates $\left(\frac{\alpha_0}{k},\frac{\alpha_1}{k},\frac{\alpha_2}{k}\right)$ satisfy $\frac{\alpha_0}{k}\geqslant 1-\frac{\mu}{k}$. Using Lemma 2.1 one easily gets

Lemma 2.2 *Let $f\in S_k^0(\tau\bigcup\tau')$ and $b_f(x_{\alpha,\tau})$, $b_f(x_{\alpha,\tau'})$ be the Béier coordinates of f with respect to τ,τ' respectively. Then $f\in C^\mu(\tau\bigcup\tau')$ if and only if the equation*

$$\sum_{|\beta|=r}\binom{r}{\beta}b_f(x_{\alpha+\beta,\tau})=\sum_{|\beta|=r}\binom{r}{\beta}b_f(x_{\alpha+\beta,\tau'})\tag{2.4}$$

holds for all $0\leqslant r\leqslant\mu$ and for any direction z such that there is a number t, $v+tz\in\tau\bigcup\tau'$. Where $\alpha=(k-r,0,0)$ and $\beta\in Z_+^3$

Therefore, $b_f(x_{\alpha,\tau'})$ and $b_f(x_{\alpha,\tau})$ can be determined each other. In other words, there is an invertible matrix A such that

$$\{b_f(x_{\alpha,\tau'})\}_{x_{\alpha,\tau'}\in P(v,\mu)}=\mathrm{A}\{b_f(x_{\alpha,\tau})\}_{x_{\alpha,\tau}\in P(v,\mu)}\tag{2.4'}$$

It is essential to represent C^μ–continuous conditions in terms of B–net representation, if spline function f is defined over two triangles with one common edge. G. Farin solved this problem in 2–dimensional case. de Boor[4] and Jia[8] considered the general n–dimensional case. Here we state Jia's succinct result for $n=2$(see [8]).

Let $\tau=\{v^0,v^1,v^2\},\tau'=\{v^0,v^1,\omega\}$ be two triangles with common edge $[v^0,v^1]$, S denote the area of triangle τ, and S_i denote the area of the striangle with vertices of τ but v^i is replaced by $\omega,i=0,1,2$.

Theorem 2.1. *Suppose that spline function f is defined on $\tau\bigcup\tau'$ by*

$$f|_\tau=\sum_{|\alpha|=k}b_{\alpha,\tau}B_{\alpha,\tau},\qquad f|_{\tau'}=\sum_{|\alpha|=k}b_{\alpha,\tau'}B_{\alpha,\tau'}.$$

Then $f\in C^\mu(\tau\bigcup\tau')$ if and only if, for all positive integers $r\leqslant\mu$ and $\alpha=(\alpha_0,\alpha_1,0)\in Z_+^3,|\alpha|=k-r$,

$$b_{\alpha+re^3,\tau'}=\sum_{|\beta|=r}\binom{r}{\beta}b_{\alpha+\beta,\tau}\left(\frac{S_0}{S}\right)^{\beta_0}\left(\frac{S_1}{S}\right)^{\beta_1}\left(\frac{S_2}{S}\right)^{\beta_2},\tag{2.5}$$

where $e^3=(0,0,1)$.

Associated with a triangulation $\triangle$, E_0 is the set of interior edges of $\triangle$. Let $e=[v^0 v^1]\in E_0$ be the common edge of two triangles $\tau=\{v^0,v^1,v^2]$ and $\tau'=\{v^0,v^1,w\}$. For integers r, j: $1\leqslant r\leqslant \mu$, $0\leqslant j\leqslant k-r$, we define the functionals on R^p by

$$\lambda_{e,j,r} b = b(x_{\alpha+re^3,\tau'}) - \sum_{|\beta|=r}\binom{r}{\beta} b(x_{\alpha+\beta,\tau})\left(\frac{S_0}{S}\right)^{\beta_0}\left(\frac{S_1}{S}\right)^{\beta_1}\left(\frac{S_2}{S}\right)^{\beta_2}, \tag{2.6}$$

where $\alpha=(k-j-r,j,0)$

It is easy to see that the support of the functional $\lambda_{e,j,r}$ is included in a diamond domain with diagonal line $\left[\frac{(k-j)v^0+jv^1}{k}, \frac{(k-j-r)v^0+(j+r)v^1}{k}\right]$ and vertices $x_{\alpha+re^3,\tau}$, and $x_{\alpha+re^3,\tau'}$ (see Figure 3).

Given $e\in E^0$, we use $\wedge_e$ to denote the set of all linear functionals $\{\lambda_{e,j,r}, 1\leqslant r\leqslant \mu, 0\leqslant j\leqslant k-r\}$. Let

$$\wedge_e^{\perp}=\{b\in R^p;\ \lambda\perp b \quad \text{for all} \quad \lambda\in\wedge_e\}.$$

Then Theorem 2.1 shows that a spline functional $f\in C^{\mu}(\tau\bigcup\tau')$ if and only if

$$b_f\in\wedge_e^{\perp}, \tag{2.7}$$

where e is the common edge of τ and τ'.

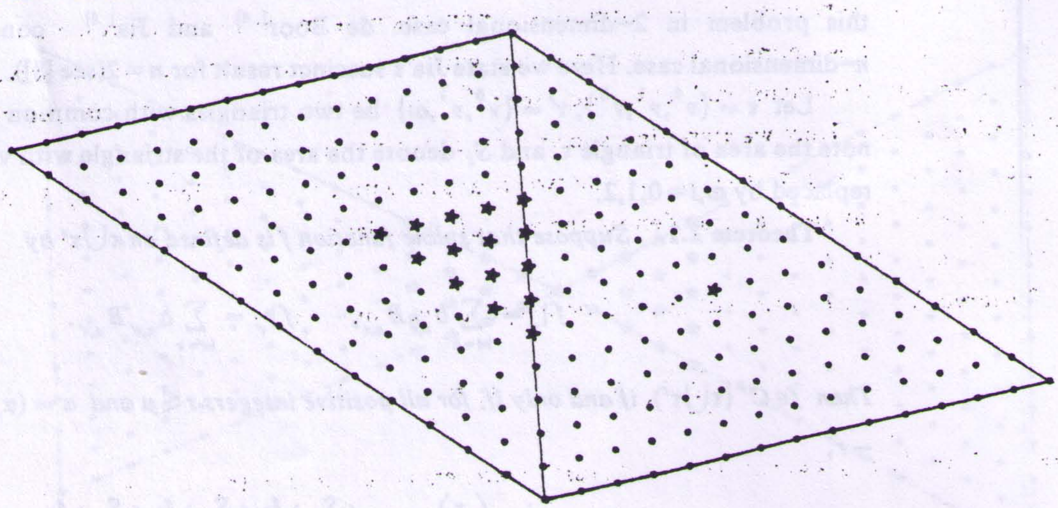


Fig.3 Support (*) of $\lambda_{e,5,4}$ (for $k=14$, $\mu=4$)

§ 3 The dimension of $S^{\mu}_{k\triangle}$

In this section we are going to prove the dimension formula for the spaces $S^{\mu}_{k\triangle}$, $k\geqslant$

As usual, $[x]$ denotes the integer part of the real number x. A B—net point $x_{\alpha,\tau}$ on the triangle $\tau=\{v^0,v^1,v^2\}$ is said to be of type I with respect to the vertex v^j if it belongs to the set

$$\left\{x_{\alpha,\tau}=\frac{1}{k}(\alpha_0 v^0+\alpha_1 v^1+\alpha_2 v^2);\ \alpha\in Z^3_+,\ |\alpha|=k,\ \alpha_j\geqslant k-\mu-[\frac{\mu}{2}]\right\}.$$

For every $v\in V$, we use X^I_v to denote the collection of all the points of type I on the partition $\triangle$ with respect to the vertex v. If the B—net point $x_{\alpha,\tau}$ is in the set

$$\left\{x_{\alpha,\tau};\ \alpha\in Z^3_+,\ |\alpha|=k,\ 0\leqslant\alpha_2\leqslant\mu,\ \mu+1<\alpha_0,\ \alpha_1<k-\mu-[\frac{\mu}{2}]\right\}.$$

then we call it a point of type II on τ with respect to edge $[v^0,v^1]$. For every edge $e\in E$, X_e denotes the set of all points of type II on $\triangle$ with respect to the edge e. Let

$$X^{IV}_\tau=\{x_{\alpha,\tau};\ \alpha_j\geqslant\mu+1,\ j=0,1,2\}$$

and call it the set of all points of type IV on τ. Besides the points of type I, II and IV, the remaining B—net points on τ are called points of type III. Clearly, the set of points of type III consists of three parts. The parts nears to the vertex v on τ is denoted by $X^{III}_\tau(v)$, and let

$$X^{III}_v=\bigcup_{\tau\in\triangle_v}X^{III}_\tau(v),\qquad X_v=X^I_v\bigcup X^{III}_v. \tag{3.0}$$

Where $\triangle_v=\{\tau\in\triangle,\ v$ is a vertex of $\tau\}$ (see Fig. 4).

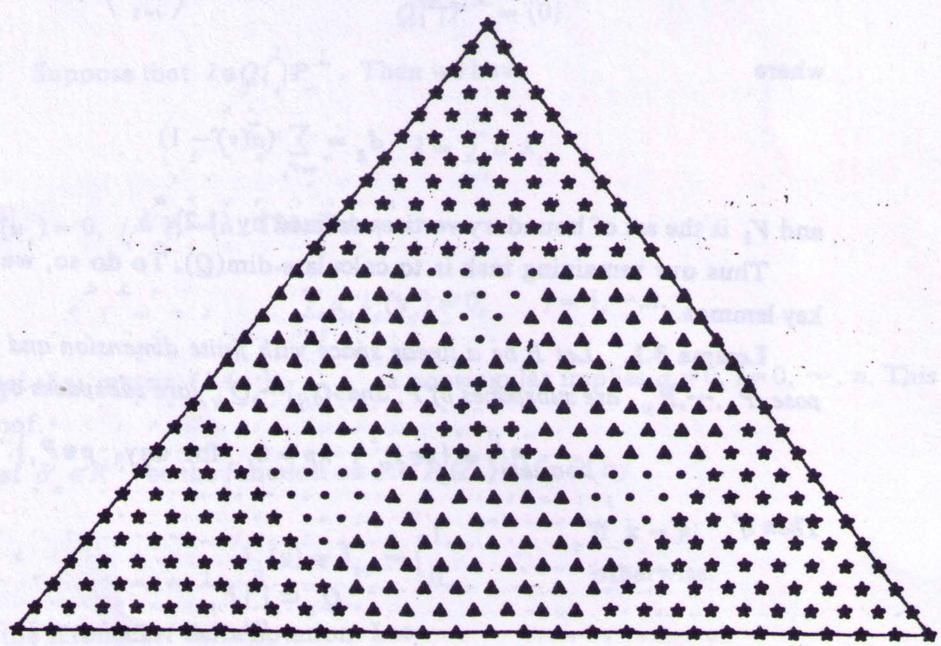


Fig.4 Classification of B—net points ($k=27$, $\mu=7$)
(I: ★; II: ▲; III: • ; IV: +)

For every $e \in E_0$, recall the definition of $\wedge_e$ in section 2. we set

$$\wedge = \bigcup_{e \in E_0} \wedge_e = \{\lambda_{e,j,r}\ \ e \in E_0,\ 0 \leqslant j \leqslant k-1,\ 1 \leqslant r \leqslant \mu\}$$

$$\wedge^{\perp} = \{b \in R^P;\ \ \lambda \perp b \quad \text{for all } \lambda \in \wedge\} \tag{3.1}$$

and

$$Q = \mathrm{span}(\wedge).$$

We denote by $d(v)$ the number of edges attached to vertex v. If $\Omega_v \subset \Omega$ denotes the standard cell on $\triangle$ with respect to the vertex v, then we define

$$\bar{d}(v): \ = \text{the number of components of } \ \Omega_v \setminus \{v\}.$$

Additionally, we use $c+1$ to denote the number of components of $R^2 \setminus \Omega$.

Having above preparation, we are ready to consider the dimension of $S^{\mu}_{k,\triangle}$. As mentioned before, any continuous spline $f \in S^0_{k,\triangle}$ is associated with its B-net representation $b_f \in R^P$. According to Theorem 2.1 and Lemma 2.2, we can conclude that $f \in S^{\mu}_{k,\triangle}$ if and only if, for every edge $e \in E_0$, b_f satisfies (2.5) and for every vertex v with $\bar{d}(v) > 1$, b_f satisfies (2.4) for arbitrary two triangles with vertex v in common. Hence we have

$$\dim(S^{\mu}_{k,\triangle}) = |P| - \dim(Q) - \left(\sum_{i=2}^{\mu+1} i\right) d_0, \tag{3.2}$$

where

$$d_0 = \sum_{v \in V_b} (\bar{d}(v) - 1)$$

and V_b is the set of boundary vertices defined by (1.2).

Thus our remaining task is to calculate $\dim(Q)$. To do so, we establish the following key lemmas.

Lemma 3.1. *Let P be a linear space with finite dimension and P^* its dual space. Suppose $P_1, \cdots, P_N$ are subspaces of P and $Q_1, \cdots, Q_N$ are subspaces of P^*. Let*

$$P_i^{\perp} = \{\lambda \in P^*;\ \ \lambda p = 0, \quad \text{for any } \ p \in P_i\}.$$

Then if

$$Q_i \subset \bigcap_{j > i} P_j^{\perp}$$

and

$$Q_i \bigcap P_i^{\perp} = \{0\} \quad \text{for} \quad i = 1, \cdots, N,$$

then

$$\dim\left(\sum_{i=1}^{N} Q_i\right) = \sum_{i=1}^{N} \dim(Q_i). \tag{3.3}$$

Proof. It is sufficient to show that $\sum_{i=1}^{N} Q_i$ is a direct sum. Suppose

$$\sum_{i=1}^{N} q_i = 0, \quad \text{for} \quad q_i \in Q_i, \quad i = 1, \cdots, N.$$

We show that $q_i = 0$ for all i.

If not, then there exists $q_n \neq 0$ for some n, such that $q_i = 0$ for all $i > n$. Then by the hypothesis, we have

$$q_n = -\sum_{i=1}^{n-1} q_i \in \bigcap_{i>n} P_i^{\perp},$$

in particular, $q_n \in P_n^{\perp}$. Note that $Q_n \bigcap P_n^{\perp} = \{0\}$, we get $q_n = 0$. This contradiction proves the conclusion.

Lemma 3.2. *If* $\{\lambda_1, \cdots, \lambda_n\}$ *spans* Q, *and for* $u_1, \cdots, u_n \in P$, *the matrix*

$$(\lambda_i(u_j))_{1 \le i,j \le n}$$

is nonsingular, then

$$Q \bigcap P^{\perp} = \{0\}.$$

Proof. Suppose that $\lambda \in Q \bigcap P^{\perp}$. Then we have

$$\lambda = \sum_{i=1}^{n} a_i \lambda_i,$$

and $\lambda(u_j) = 0, \; j = 1, \cdots, n$. Therefore,

$$\sum_{i=1}^{n} a_i \lambda_i(u_j) = 0, \qquad j = 1, \cdots, n.$$

The fact that matrix $(\lambda_i(u_j))_{1 \le i,j \le n}$ is nonsingular implies $a_i = 0, \; i = 0, \cdots, n$. This finishes the proof.

Let $\delta_x \in R^P$ be the function on $P = P_k(\triangle)$ defined by

$$\delta_x(y) = \delta_{xy} = \begin{cases} 1, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

$\delta_{x,y}$ is the Kronecker delta notation. Let

$$P_v = \text{span}\{\delta_x; \; x \in X_v\} \quad \text{for} \quad v \in V$$

$$P_e = \text{span}\{\delta_x; \; x \in X_e\} \quad \text{for} \quad e \in E_0.$$

By the definition of X_v and X_e, we have

$$X_e \bigcap X_v = \emptyset, \quad \text{for any} \quad e \in E_0, \quad v \in V. \tag{3.4}$$

Since $k \geqslant 3\mu + 2$, we have

$$X_v \bigcap X_{v'} = \emptyset, \quad \text{for} \quad v \neq v'$$

$$X_e \bigcap X_{e'} = \emptyset, \quad \text{for} \quad e \neq e'. \tag{3.5}$$

Let

$$\wedge_v = \{\lambda \in \wedge;\ \operatorname{supp}\lambda \subset X_v\}$$

$$\wedge_e = \{\lambda \in \wedge;\ (\operatorname{supp}\lambda) \bigcap X_e \neq \emptyset\}$$

and

$$Q_v = \operatorname{span}(\wedge_v), \qquad Q_e = \operatorname{span}(\wedge_e). \tag{3.6}$$

Then

$$Q = \sum_{v \in V} Q_v + \sum_{e \in E_0} Q_e. \tag{3.7}$$

Note that $k \geqslant 3\mu+2$, we get from (3.4) and (3.5) that

$$Q_v \subset P_e^{\perp} \quad \text{for any} \quad e \in E_0,\ v \in V,$$

$$Q_v \subset P_{v'}^{\perp} \quad \text{for any} \quad v \neq v',\ Q_e \subset P_{e'}^{\perp} \quad \text{for any} \quad e \neq e'. \tag{3.8}$$

From Lemma 3.2 we have also

$$Q_v \bigcap P_v^{\perp} = \{0\}, \text{ for any } v \in V. \tag{3.9}$$

If we arrange the elements of $V \bigcup E_0$ in an order such that the edges are after the vertices, then, by Lemma 3.1, once we prove

$$P_e^{\perp} \bigcap Q_e = \{0\} \qquad \text{for all} \qquad e \in E_0, \tag{3.10}$$

we have

$$\dim(Q) = \sum_{v \in V} \dim(Q_v) + \sum_{e \in E_0} \dim(Q_e). \tag{3.11}$$

Next we are going to prove (3.10). The following result which is essential in proving (3.10) can be derived from Theorem 10.1 of Karlin ([9],Chapter 8).

Lemma 3.3. *Let*

$$A_{n,n} = \begin{bmatrix} \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{2!} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{n!} \end{bmatrix}.$$

Then $\det(A_{n,n}) \neq 0$.

Lemma 3.4. *If* $k \geqslant 3\mu+2$, *then* (3.10) *is true for any* $e \in E_0$.

Proof. Suppose that $\tau = \{v^0, v^1, v^2\}$ and $\tau' = \{v^0, v^1, w\}$ are two triangles with the edge $e = [v^0, v^1]$ in common. Then the support of every functional $\lambda_{e,ij}$ is a one-point subset in τ'. S_0 and S_1 denote the areas of the triangles $\{w, v^1, v^2\}$ and $\{v^0, w, v^2\}$, respectively. We divide the proof into two cases.

Case 1. Both S_0 and S_1 are not zero.

We use $\bar{\mu}$ to denote $[(\mu+1)/2]$ and let $\underline{\mu} = [\mu/2]$. For $1 \leqslant m \leqslant \bar{\mu}$, let

$$X_{2m-1} = X_m(v^0)$$
$$= \{x_{\alpha,\tau} \in X_e;\ \alpha = (k-\mu-\underline{\mu}-m, \mu+t, \underline{\mu}+m-t),\ 1 \leqslant t \leqslant \bar{\mu}-m+1\},$$
$$X_{2m} = X_m(v^1)$$
$$= \{x_{\alpha,\tau} \in X_e;\ \alpha = (\mu+t, k-\mu-\underline{\mu}-m, \underline{\mu}+m-t),\ 1 \leqslant t \leqslant \bar{\mu}-m+1\}. \quad (3.12)$$

and define

$$X_{2\bar{\mu}+1} = X_e \bigcap (\tau' \backslash e). \quad (3.13)$$

We figure the points of X_{2m} in Fig.5. Let

$$P_i = \mathrm{span}\{\delta_x;\ x \in X_i\}, \quad i = 1, 2, \cdots, 2\bar{\mu}+1.$$

and

$$\wedge_{2\bar{\mu}+1} = \{\lambda \in \wedge_e;\ \mathrm{span}\lambda \bigcap (\tau' \backslash e) \subset X_{2\bar{\mu}+1}\}.$$

For $i = 1, \cdots, 2\bar{\mu}$, we define $\wedge_i$ inductively

$$\wedge_i = \{\lambda \in \wedge_e;\ \lambda \notin \wedge_j \text{ for } j > i, \text{ and } \mathrm{supp}\,\lambda \supset X_i\}. \quad (3.14)$$

So, if denote

$$Q_i = \mathrm{span}(\wedge_i),\ i = 1, 2, \cdots, 2\bar{\mu}+1.$$

then

$$Q_e = \sum_{i=1}^{2\bar{\mu}+1} Q_i.$$

Noticing that $k \geqslant 3\mu+2$, we get

$$(\mathrm{supp}\wedge_i)\bigcap X_j = \phi \quad \text{for } j > i. \tag{3.15}$$

Thus

$$Q_i \subset P_j^{\perp} \quad \text{for} \quad j > i \ . \tag{3.16}$$

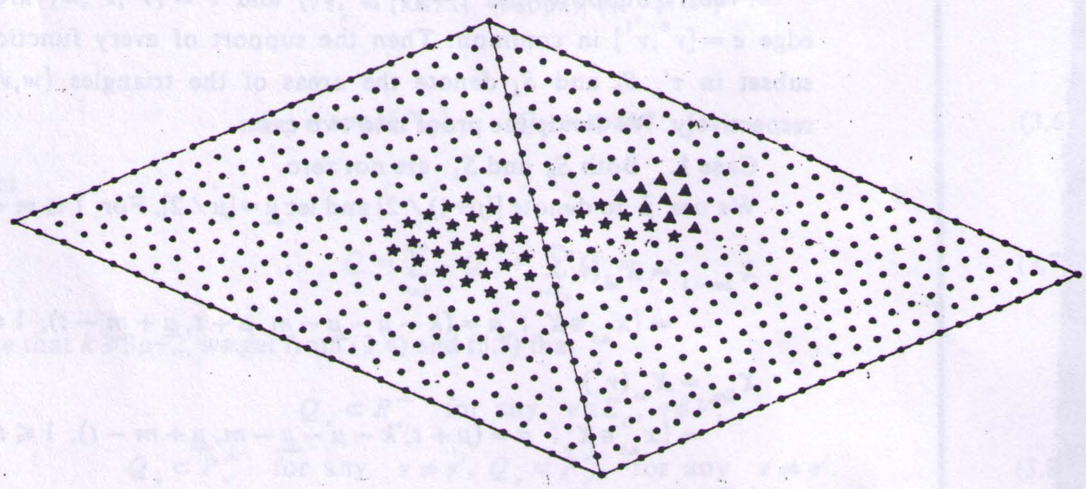


Fig.5 The tips(▲) of the support diamonds of $\lambda \in \wedge_{2m}$ and the set X_{2m} (★ points) for ($1 \leqslant m \leqslant 4$; $\mu = 7$, $k = 24$)

If we set $s_i := \bar{\mu} + 1 - [(i+1)/2]$, then

$$|\wedge_i| = s_i, \ 1 \leqslant i \leqslant 2\bar{\mu} + 1.$$

Given $\lambda_1, \cdots, \lambda_{s_i}$ are s_i elements of $\wedge_i$, and assume that $a_1\lambda_1 + \cdots + a_{s_i}\lambda_{s_i} \in P_i^{\perp}$, then from $\lambda(\delta_x) = 0$, for any $x \in X_i$, one gets the following system of linear equations

$$\mathbf{B}_{s_i s_i}\mathbf{a} = 0, \tag{3.17}$$

where $\mathbf{B}_{s_i s_i}$ is an $s_i \times s_i$ matrix, while $\mathbf{a}$ is a column vector and its transpose $\mathbf{a}^t = (a_1, \cdots, a_{s_i}) \in R^{s_i}$.

Since $S_0 \neq 0$ and $S_i \neq 0$, we get by calculating that

$$\mathrm{set}(\mathbf{B}_{s_i s_i}) = M\det(\mathbf{A}_{s_i s_i}) \tag{3.18}$$

where M is a nonzero constant and $\Lambda_{e_i e_i}$ is defined as in Lemma 3.3. Hence the system of linear equations (3.17) has only zero solution. Therefore, we obtain

$$Q_i \bigcap P_i^{\perp} = \{0\}, \text{ for } i = 1, \cdots, 2\bar{\mu}. \tag{3.19}$$

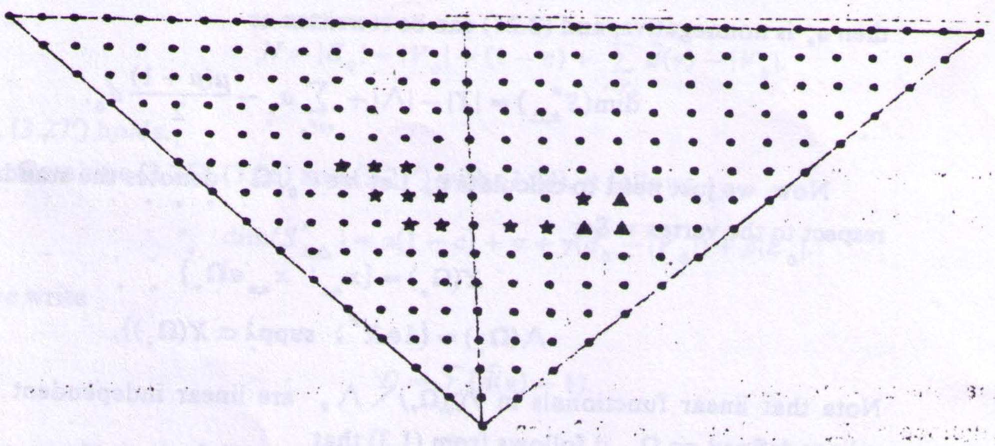


Fig.6 The tips (▲) of the support of $\wedge_{2m-1}$ and the sets X_{2m-1} (★) for $S_0 = 0$, but $\wedge_{2m} = \phi$ ($k = 14, \mu = 4, m = 1, 2$)

Moreover, the fact that $\mathrm{supp}\lambda \bigcap (\tau \diagdown e)$ is a one-point set for every $\lambda \in \wedge_{2\bar{\mu}+1}$ implies

$$Q_{2\bar{\mu}+1} \bigcap P_{2\bar{\mu}+1}^{\perp} = \{0\}. \tag{3.20}$$

Using Lemma 3.1, we get that the sum in the expression

$$Q = \sum_{i=1}^{2\bar{\mu}+1} Q_i$$

is a direct sum. Hence, if $q = \sum q_i \in P_i^{\perp}$, $q_i \in Q_i$, then $q_i \in P_i^{\perp}$. So, (3.19) and (3.20) implies $q = 0$. So (3.10) is true in this case.

Case2. $S_0 = 0$ or $S_1 = 0$.

Without loss of any generality, we assume $S_0 = 0$. Then $S_1 \neq 0$. Notice that the sets $\wedge_i$ defined by (3.14) are empty when $i = 2m, m = 1, \cdots, \bar{\mu}$ (see Fig.6), the proof given above is still valid in the present situatoin. So we complete the proof.

Combining (3.2) with (3.11), we have

$$\dim(S^{\mu}_{k\triangle}) = |X| - \left(\sum_{j=2}^{\mu+1} j\right) d_0 - \left(\sum_{v \in V} \dim(Q_v) + \sum_{e \in E_0} \dim(Q_e)\right). \tag{3.21}$$

Evidently, we have

$$\dim(Q_e) = |\wedge_e| \text{ for every } e \in E_0,$$

$$\dim(Q_v) = |\wedge_v| \text{ for every } v \in V_b. \quad (3.23)$$

For $v \in V_0$, let

$$a_v := |\wedge_v| - \dim(Q_v). \quad (3.23)$$

then a_v is nonnegative, and (3.21) can be rewritten as

$$\dim(S^{\mu}_{k,\triangle}) = |X| - |\wedge| + \sum_{v \in V_0} a_v - \frac{\mu(\mu+1)}{2} d_0. \quad (3.24)$$

Now, we just need to calculate a_v. Let $v \in V_0, \Omega_v$ denotes the standard cell on $\triangle$ with respect to the vertex v. Set

$$X(\Omega_v) = \{x_{\alpha,\varepsilon};\ x_{\alpha,\varepsilon} \in \Omega_v\}$$

$$\wedge(\Omega_v) = \{\lambda \in R^P;\ \text{supp}\lambda \subset X(\Omega_v)\}.$$

Note that linear functionals in $\wedge(\Omega_v) \setminus \wedge_v$ are linear independent if we consider the splines defined on Ω_v, it follows from (1.3) that

$$\begin{aligned} a_v &= \dim(S^{\mu}_{k,\triangle}(\Omega_v)) - (|X(\Omega_v)| - |\wedge(\Omega_v)|) \\ &= (\alpha + \beta d(v) - \gamma - \sigma_v) - (\alpha - (k+1) - k\mu + \frac{\mu(\mu-1)}{2})d(v) \\ &= \frac{(\mu+1)(\mu+2)}{2} + \sigma_v. \end{aligned}$$

Where α, β, γ and σ_v are defined as in (1.4). Moreover, we have

$$|X| = \alpha N - (k+1)|E_0| - d_0,$$

$$|\wedge| = \sum_{j=1}^{\mu} (k+1-j)|E_0|, \quad (3.26)$$

where N is the number of triangles in the partition $\triangle$. If Ω is a connected domain, then Euler's formula for a planar graph yields

$$N = |E| - |V| + (1-c), \quad (3.27)$$

where $c+1$ denotes the number of connected components of $R^2 \setminus \Omega$. Furthermore, (3.27) can be rewritten as

$$N = |E_0| - |V_0| + (1-c) + d_0 \quad (3.27')$$

where d_0 is defined as in (3.2). To verify it, one only needs to consider the graph $G = (V_b, E_b)$ which is composed of boundary vertices V_b and boundary edges E_b of partition $\triangle$. Recall an elementary theorem in graph theory ([6], Theorem 2.1),

$$\sum_{v\in V_b} d(v) = 2|E_b|,$$

where $d(v)$ is the degree of vertex v in graph G. We have

$$\sum_{v\in V_b} 2\bar{d}(v) = 2|E_b|.$$

Substitute it into (3.27), we get

$$N = |E_0| - |V_0| + (1-c) + \sum_{v\in V_b} \bar{d}(v) - |V_b|.$$

i.e., (3.27′) holds.

Combine (3.25), (3.26) and (3.27′) with (3.24), it follows

$$\dim(S^{\mu}_{k,\triangle}) = \alpha(1-c) + \sigma + \gamma(d_0 - |V_0|) + \beta|E_0|.$$

If we write

$$D = \sum_{v\in V} (\bar{d}(v) - 1). \tag{3.28}$$

Then we obtain the following main result:

Theorem 3.1. *Suppose $\triangle$ is a trianglution of Ω. Let $c+1$ denote the number of connected components of $R^2\setminus\Omega$, and $k\geqslant 3\mu+2$. Then the dimension of space $S^{\mu}_{k,\triangle}$ is*

$$\dim(S^{\mu}_{k,\triangle}) = \alpha(1-c) + \beta|E_0| + \gamma D + \sigma, \tag{3.29}$$

where α,β,γ and σ and defined by (1.6) and (1.4), while D is given by (3.28).

From Theorem 3.1, one gets immediately the following:

Corollary 3.1. *If Ω is a polygon, and $k\geqslant 3\mu+2$, then the equality holds in (1.4) of Theorem S_2. Moreover, when Ω is a complex connected domain, (1.4) may not be true.*

In fact, if Ω is a polygon, then $c=0$, $d_0=0$, hence (3.29) becomes

$$\dim(S^{\mu}_{k,\triangle}) = \alpha + \beta|E_0| - \gamma|V_0| + \sigma.$$

If we choose a triangulation for a not simply connected domain, such that $c>0$, and $d_0=1$, then (3.29) shows

$$\dim(S^{\mu}_{k,\triangle}) < \alpha + \beta|E_0| - \gamma|V_0| + \sigma.$$

§ 4. Basis for the space $S^{\mu}_{k,\triangle}$

In this section, we are going to construct a locally supported basis for the space $S^{\mu}_{k,\triangle}$ in the case $k\geqslant 3\mu+2$. Furthermore, the basis functions constructed are minimally supported.

We say Ω is a standard cell if it is triagulated with precisely one interior vertex v such that every boundary vertex is connected to v by a interior edge. We state the definition of minimally supported basis as following (see [2]).

Definition 4.1. A basis of $S^{\mu}_{k,\triangle}$ is called a minimally supported basis, if the support of each spline in the basis is a subset of a standard cell.

As before, we use $P = P_k(\triangle)$ to denote the set of all B-net points on partition $\triangle$, for $s \in S^{0}_{k,\triangle}$, b_s is its B-net representation. Since the B-net mapping **b** gives a one-to-one correspondence between $\wedge^{\perp}$ and $S^{\mu}_{k,\triangle}$, it suffices to construct a minimally supported basis for $\wedge$. We need the following definition.

Definition 4.2. A set $X \subset P_k(\triangle)$ is said to be a determining set for $S^{\mu}_{k,\triangle}$ if

$$s \in S^{\mu}_{k,\triangle}: \ b_s(\delta_x) = 0, \quad x \in X, \Rightarrow b_s \equiv 0.$$

A determining set X is a minimal determining set if there exists no determining set with fewer elements.

Clearly, a set $X \subset P_k(\triangle)$ is a minimal determining set for $S^{\mu}_{k,\triangle}$ if and only if X is a determining set of $S^{\mu}_{k,\triangle}$ and $|X| = \dim(S^{\mu}_{k,\triangle})$.

To find a minimal determining set X for $S^{\mu}_{k,\triangle}$, we concerned with the B-net pointset X_v defined by (3.0) first. Recall the definition of $\wedge_v$ and Q_v by (3.6), we define

$$Y_v = \operatorname{supp}(\wedge_v)$$
$$P = \operatorname{span}\{\delta_x \in R^m; \ x \in X\},$$

where $m = |X_v|$. because Q_v is a linear space of finite dimension. we choose $\lambda_1, \cdots, \lambda_n \in \wedge_v$ such that they form a basis for Q_v. Then, for any $b \in P_v$, $b = (b_1, \cdots, b_m)$,

$$\lambda_i(b) = 0, \ \text{for } i = 1, \cdots, n$$

or say

$$\sum_{j=1}^{m} \lambda_{ij} b_j = 0, \qquad i = 1, \cdots, n \tag{4.1}$$

where $\lambda_i = (\lambda_{i1}, \cdots, \lambda_{im})$. The coefficients matrix of (4.1) are full row rank, therefore, there exists a subset $\{j_1, \cdots, j_n\}$ of set $\{1, \cdots, m\}$ such that

$$A := \det \begin{bmatrix} \lambda_{1j_1} & \lambda_{1j_2} & \cdots & \lambda_{1j_n} \\ \lambda_{2j_1} & \lambda_{2j_2} & \cdots & \lambda_{2j_n} \\ \vdots & \cdots & \cdots & \vdots \\ \lambda_{nj_1} & \lambda_{nj_2} & \cdots & \lambda_{nj_n} \end{bmatrix}$$

$$= \max_{\{i_1,\cdots,i_n\} \subset \{1,\cdots,N\}} \det \begin{bmatrix} \lambda_{1i_1} & \cdots & \lambda_{1i_n} \\ \lambda_{2i_1} & \cdots & \lambda_{2i_n} \\ \vdots & & \vdots \\ \lambda_{ni_1} & \cdots & \lambda_{ni_n} \end{bmatrix}$$

$$=: \max_i A_i$$

Using Gram's rule, we solve the system of linear equations (4.1), and obtain

$$|b_{jl}| \leqslant \sum_{\substack{i \neq jl \\ i=1,\cdots,n}} |b_i| \frac{A_i}{A} \leqslant \sum_{\substack{i \neq jl \\ i=1,\cdots,n}} |b_i|. \tag{4.2}$$

Corresponding to $\{b_{jl}\}_{l=1}^{n}$, the B-net pointset $\{x_{jl}\}_{l=1}^{n}$ is denoted by X_0, and let $\tilde{X}_v = X_v \setminus X_0$.

For every interior edge e, X_e denotes the set of B-net points of type II on the neighboring triangles sharing e. Let

$$\tilde{X}_e = X_e \setminus \bigcup_{i=1}^{2\bar{\mu}+1} X_i,$$

where $X_i\,(i=1,\cdots,2\bar{\mu}+1)$ are defined by (3.12) and (3.13).

Now we are ready to state the following theorem.

Theorem 4.1. *Let A_k^μ be the set*

$$A_k^\mu = (\bigcup_{v\in V} \tilde{X}_v) \bigcup (\bigcup_{e\in E_0} \tilde{X}_e) \bigcup (X \setminus (\bigcup_{v\in V} X_v \bigcup_{e\in E_0} X_e)).$$

Then A_k^μ is a minimal determining set for $S_{k,\triangle}^\mu$.

In fact, according to the proof of Theorem 3.1, we have

$$|A_k^\mu| = \dim(S_{k,\triangle}^\mu).$$

On the other hand, if we arrange the elements of $V \bigcup E_0$ in an order such that the edges are after the vertices, then for $b_f \in R^P$, we can write the smoothness conditions, $b_f \in \wedge^\perp$ in the form of

$$\wedge b_f = 0, \tag{4.3}$$

where

$$\wedge = \begin{bmatrix} (\wedge_v)_{v\in V} & 0 \\ 0 & (\wedge_e)_{e\in E_0} \end{bmatrix}$$

Furthermore, (4.3) can be rewritten as

$$\mathbf{A}\mathbf{b}_1 = \mathbf{B}\mathbf{b}_2,$$

where $\mathbf{b}_2 = (b_f(x_{\alpha,\tau}))_{x_{\alpha,\tau}\in A_k^\mu}$, $\mathbf{b}_1 = (b_f(x_{\alpha,\tau}))_{x_{\alpha,\tau}\in P\setminus A_k^\mu}$ and $\mathbf{A}$ is a full column rank block lower–triangular matrix. Therefore, A_k^μ is a minimal determining set.

Denote

$$A_k^\mu = \{x_i\}_{i=1}^d,$$

where $d = |A_k^\mu| = \dim(S_{k,\triangle}^\mu)$. Then we define $b_i \in R^P$ $(i = 1,2,\cdots,d)$ by

$$b_i(x_j) = \delta_{ij} \quad for \ x_j \in A_k^\mu;$$

On the other points $x_{\alpha,\tau} \in P$, $b_i(x_{\alpha,\tau})$ is determined by the C^μ–continuous conditions (4.3).

Clearly, $b_i(i = 1,2,\cdots,d)$ is uniquely determined in this way. Let s_i be the spline functions in $S_{k,\triangle}^0$ such that their B–net representations are $b_i(i = 1,2,\cdots,d)$. Then we have

Theorem 4.2. $\{s_i\}_{i=1}^d$ *is a minimally supported basis for the spaces* $S_{k,\triangle}^\mu$.

Proof. We only need to show that $\{b_i\}_{i=1}^d$ is a basis for the space $\wedge^\perp$ and that for each b_i, it is minimally supported. Obviously, b_i satisfies the smoothness conditions and $\{b_i\}$ is linear independent. In view of that fact

$$\dim(\wedge^\perp) = \dim(S_{k,\triangle}^\mu) = d,$$

we know that $\{b_i\}_{i=1}^d$ is a basis of $\wedge$. Finally, we are going to show that b_i is minimally supported. There are three possible cases.

If $x_i \in X_v$, i.e. x_i is a point of type *I* or type *III* of vertex v, then the smoothness conditions (2.7) can be accommodated by assigning appropriate values to the B–net values $b(x_{\alpha,\tau})$ in X_v and in $X_m(v)$, which is defined by (3.12), for each interior edge e emanating from v. Thus the support of such a basis functions is included in Ω, the standard cell with interior vertex v.

If $x_i \in X_e$, $e \in E_0$, then the smoothness conditions (2.7) can be accommodated by assigning appropriate values to $b_i(x_{\alpha,\tau})$ in X_e. In this case, its support is included in the triangles containing the relevant edge e.

If $x_i \in X_e$, $e \in E_b$, or x_i is a point of type *IV*, then we can choose $b_i(x_{\alpha,\tau}) = 0$ for all $x_{\alpha,\tau} \neq x_i$. This means the support of such basis function is included in the triangles which contains the B–net point x_i. Thus we are done.

Remarks 1.We have not been able to extend the method of proof used here to lower degree spline spaces, for example to $S_{3\mu+1}^\mu(\triangle)$. But we conjecture that the dimension formula (3.29) is true also. In the case $\mu = 1$, $k = 4$, the dimension of $S_4^1(\triangle)$ has been settled in [3]. For $\mu \geqslant 2$, the question is still open.

2. de Boor [5] proved that the space $S^{\mu}_{k,\triangle}$ has the full approximation order $r = k+1$ using the duality of the space $S^{\mu}_{k,\triangle}$ whenever $k \geqslant 3\mu+2$. We considered this question in [7] by constructing an approximation scheme using the local basis of $S^{\mu}_{k,\triangle}$, $k \geqslant 3\mu + 2$ which was constructed in section 4

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