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Some smoothness conditions and conformality conditions for bivariate quartic and quintic splines

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Abstract. This paper is concerned with a study of some new formulations of smoothness conditions and conformality conditions for multivariate splines in terms of B-net representation. In the bivariate setting, a group of new parameters of bivariate quartic and quintic polynomials over a planar simplex is introduced, new formulations of smoothness conditions of bivariate quartic C^1 splines and quintic C^2 splines are given, and the conformality conditions of bivariate quartic C^1 splines are simplified.

1 Introduction

The Bernstein–Bézier method (B-form, B-net) plays an important role in the study of both curve fitting and multivariate spline approximation. The B-net was initiated from Bernstein polynomials. In the late fifties and the early sixties, de Casteljau and Bézier applied Bernstein polynomials in the study of curve fitting by using triangular patches and rectangular patches. In 1980, Farin [1] first used the B-net method in the study of bivariate splines. More details about the Bernstein–Bézier patches can be found in [2,3].

The B-net is widely applied in the study of either the dimension and basis or the approximation property of multivariate spline spaces (see [4]–[9] for example). Some applications of the B-net method in the study of multivariate splines can be found in [10]. In this paper, we study formulations of smoothness conditions of multivariate splines determined by some interpolation conditions at the vertices and some B-net domain points. In particular, we give a simplified formulation of smoothness conditions for bivariate quartic and quintic splines and simplified conformality conditions for bivariate quartic splines. The paper is organized as follows. In Sect. 2, we give a brief review of the B-net representation of spline functions and recall some formulations of smoothness conditions for multivariate splines. In Sect. 3, we discuss smoothness conditions for bivariate C^1 quartic and C^2 quintic splines. Conformality conditions for bivariate quartic C^1 splines are discussed in Sect. 4.

2 B-net representation of multivariate splines

As usual, let $\mathbf{R}$ be the set of all real numbers and $\mathbf{Z}_+$ the set of nonnegative integers. Thus $\mathbf{R}^n$ denotes n-dimensional Euclidean space and $\mathbf{Z}_+^n$ can be used as a multi-index set, while $\pi_k := \pi_k(\mathbf{R}^n)$ is the space of all polynomials of (total) degree $\leq k$ in n variables. Let $\delta = [v_0, v_1, \ldots, v_n]$ be a proper n-dimensional simplex with vertices $v_0,\ v_1,\ \ldots,\ v_n\ \in\ \mathbf{R}^n$. Then for any $x \in \mathbf{R}^n$, we have

$$x = \xi_0 v_0 + \xi_1 v_1 + \cdots + \xi_n v_n \quad \text{with} \quad \xi_0 + \xi_1 + \cdots + \xi_n = 1.$$

The $(n+1)$-tuple $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$ defines the barycentric coordinates of x with respect to the simplex δ. For $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^{n+1}$, the length of α is defined by $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n$, and the factorial $\alpha!$ is defined as $\alpha_0! \ldots \alpha_n!$. We define the Bernstein polynomial $B_{\alpha,\delta}$ as

$$B_{\alpha,\delta}(x) := \binom{|\alpha|}{\alpha} \xi^\alpha,$$

where $\xi^\alpha := \xi_0^{\alpha_0} \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$ and

$$\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha_0! \alpha_1! \ldots \alpha_n!}.$$

In addition, the domain points $x_{\alpha,\delta}$ on δ are defined by

$$x_{\alpha,\delta} := \frac{(\alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_n v_n)}{k}, \quad |\alpha| = k.$$

It is well-known that any polynomial $p \in \pi_k$ can be written in a unique way as

$$p = \sum_{|\alpha|=k} b_{\alpha,\delta} B_{\alpha,\delta},$$

where $b_{\alpha,\delta}$ is called the B-net ordinate of p with respect to δ. This gives rise to a map $b : x_{\alpha,\delta} \mapsto b_{\alpha,\delta}$, $|\alpha| = k$. Such a map b is called the B-net representation of p with respect to δ.

Let $\triangle$ be a triangulation of a polygonal domain in $\mathbf{R}^n$ and $S_k^r(\triangle)$ the linear space of piecewise polynomial functions (splines) with total degree $\leq k$ and smoothness order r on $\triangle$. Assume $s \in S_k^0(\triangle)$. Then on each simplex $\delta \in \triangle$, s agrees with some polynomial $p \in \pi_k$. Thus, we have

$$s|_\delta = \sum_{|\alpha|=k} b_{\alpha,\delta} B_{\alpha,\delta}.$$

Let X denote the set of all (domain) points $x_{\alpha,\delta}$. Then a map can be defined as follows:

$$b_s : x_{\alpha,\delta} \mapsto b_{\alpha,\delta}, \quad |\alpha| = k, \delta \in \triangle.$$

Such a map b_s is called the B-net representation of the spline function s.

Let $\delta = [v_0, v_1, \ldots, v_n]$ and $\tilde{\delta} = [v_0, v_1, \ldots, \tilde{v}_n]$ be two n-dimensional simplices with a common $(n-1)$-dimensional face $[v_0, v_1, \ldots, v_{n-1}]$ and denote $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$, $i = 0, 1, \ldots, n$, and $\tilde{v}_n = (\tilde{v}_{n1}, \tilde{v}_{n2}, \ldots, \tilde{v}_{nn})$. Then the oriented volume of the simplex δ is

$$V := \mathrm{vol}[v_0, v_1, \ldots, v_n] = \frac{1}{n!} \begin{vmatrix} 1 & v_{01} & v_{02} & \ldots & v_{0n} \\ 1 & v_{11} & v_{12} & \ldots & v_{1n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & v_{n1} & v_{n2} & \ldots & v_{nn} \end{vmatrix}. \tag{1}$$

If we set $\hat{v}_i = \tilde{v}_n$, then we will denote the oriented volume of the simplex $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$ by

$$V_i := \mathrm{vol}[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n] = \frac{1}{n!} \begin{vmatrix} 1 & v_{01} & \ldots & v_{0n} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & v_{i-11} & \ldots & v_{i-1n} \\ 1 & \tilde{v}_{n1} & \ldots & \tilde{v}_{nn} \\ 1 & v_{i+11} & \ldots & v_{i+1n} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & v_{n1} & \ldots & v_{nn} \end{vmatrix}. \tag{2}$$

The following result, which describes C^r-smoothness conditions on a spline function s in terms of its B-net representation, is from [11] (see also [12,13]).

Theorem 1 *Suppose that the piecewise polynomial function s is defined on $\delta \cup \tilde{\delta}$ by*

$$s|_\delta = \sum_{|\alpha|=k} b_{\alpha,\delta} B_{\alpha,\delta},$$

$$s|_{\tilde{\delta}} = \sum_{|\alpha|=k} b_{\alpha,\tilde{\delta}} B_{\alpha,\tilde{\delta}}.$$

Then $s \in C^r(\delta \cup \tilde{\delta})$ *if and only if, for all positive integers* $\ell \leq r$ *and* $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, 0) \in \mathbf{Z}_+^{n+1}$ *with* $|\alpha| = k - \ell$,

$$b_{\alpha+\ell \mathbf{e}^{n+1},\tilde{\delta}} = \sum_{|\beta|=\ell} \binom{\ell}{\beta} b_{\alpha+\beta,\delta} \frac{V_0^{\beta_0} \ldots V_n^{\beta_n}}{V^{|\beta|}}, \tag{3}$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$ *and* $\mathbf{e}^{n+1} = (0, \ldots, 0, 1)$ *are in* $\mathbf{Z}_+^{n+1}$.

For the purpose of studying stability of the space of bivariate splines with smoothness order r and total degree $k \geq 3r+2$, the following new formulation of smoothness conditions is derived in [14]. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$. We use standard multi-index notations. We say that $\gamma \leq \alpha$ if and only if $\gamma_i \leq \alpha_i$ for $i = 1, \ldots, n$. For $\alpha, \gamma \in \mathbf{Z}_+^n$ with $\gamma \leq \alpha$, we have

$$\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n},$$
$$\alpha! = \alpha_1! \ldots \alpha_n!,$$
$$(\alpha - \gamma)! = (\alpha_1 - \gamma_1)! \ldots (\alpha_n - \gamma_n)!$$

and

$$\binom{\alpha}{\gamma} = \binom{\alpha_1}{\gamma_1} \ldots \binom{\alpha_n}{\gamma_n} = \frac{\alpha!}{(\alpha-\gamma)!\gamma!}.$$

Let

$$C_{\alpha,\delta} := \sum_{\gamma \leq \alpha} (-1)^{|\alpha-\gamma|} \binom{\alpha}{\gamma} b(x_{\gamma,\delta}),$$

where

$$x_{\alpha,\delta} := \frac{(k - \alpha_1 - \cdots - \alpha_n)v_0 + \alpha_1 v_1 + \cdots + \alpha_n v_n}{k}.$$

We have the following formulation of smoothness conditions for multivariate spline functions [14].

Theorem 2 *A spline function* $s \in C^0(\delta \cup \tilde{\delta})$ *is of smoothness order* r *if and only if the corresponding terms* $\{C_{\alpha,\delta}\}$ *and* $\{C_{\alpha,\tilde{\delta}}\}$ *satisfy the condition:*

$$C_{\alpha,\tilde{\delta}} = \sum_{|\gamma^-| \leq \alpha_n} C_{(\alpha+\gamma^- - |\gamma^-|\mathbf{e}^n),\delta} \frac{\alpha_n!}{\gamma^-!(\alpha_n - |\gamma^-|!)} \frac{V_1^{\gamma_1} \ldots V_{n-1}^{\gamma_{n-1}} V_n^{\alpha_n - |\gamma^-|}}{V^{\alpha_n}} \tag{4}$$

for $1 \leq \alpha_n \leq r$, $\alpha, \gamma \in \mathbf{Z}_+^n$ *with* $|\alpha| = k$, *where* $\gamma^- := (\gamma_1, \ldots, \gamma_{n-1}, 0) \in \mathbf{Z}_+^n$.

An application of this smoothness formulation can be found in [6].

3 Smoothness conditions of quartic and quintic splines

In this section and the next, we restrict ourselves to the two-dimensional setting. Let $\delta_1 = [v_0, v_1, v_2]$ be a planar non-degenerate simplex with vertices $v_i = (x_i, y_i) \in \mathbf{R}^2,\ i = 0, 1, 2$. Let $\xi = (\xi_0, \xi_1, \xi_2)$ be the barycentric coordinates of $x \in \mathbf{R}^2$ with respect to the simplex δ_1. Set

$$\begin{aligned} \lambda_i &:= y_j - y_k, \\ \mu_i &:= -(x_j - x_k), \\ \nu_i &:= x_j y_k - y_j x_k, \end{aligned}$$

with (i, j, k) a cycling of the subscripts in cyclic order $0 \to 1 \to 2 \to 0$. Clearly, the following equalities hold:

$$\begin{cases} \lambda_0 + \lambda_1 + \lambda_2 = 0 \\ \mu_0 + \mu_1 + \mu_2 = 0 \\ \nu_0 + \nu_1 + \nu_2 = 0. \end{cases} \tag{5}$$

Also the barycentric coordinates can be expressed as

$$\begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \frac{1}{A^{(1)}} \begin{bmatrix} \nu_0 & \lambda_0 & \mu_0 \\ \nu_1 & \lambda_1 & \mu_1 \\ \nu_2 & \lambda_2 & \mu_2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}, \tag{6}$$

where

$$A^{(1)} := 2\ \text{area}[v_0, v_1, v_2] = \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}$$

is the oriented area of the simplex δ_1 as given in (1).

For a bivariate polynomial p with total degree k with B-net representation $b_\alpha^{(1)}$, i.e.,

$$p = \sum_{|\alpha|=k} b_\alpha^{(1)} B_{\alpha,\delta_1} = \sum_{|\alpha|=k} b^{(1)}_{\alpha_0\alpha_1\alpha_2} B_{\alpha,\delta_1}, \tag{7}$$

we have the following

$$\begin{aligned} \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \xi_0}{\partial x} & \frac{\partial \xi_1}{\partial x} \\ \frac{\partial \xi_0}{\partial y} & \frac{\partial \xi_1}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial \xi_0} \\ \frac{\partial p}{\partial \xi_1} \end{bmatrix} \\ &= \frac{1}{A^{(1)}} \begin{bmatrix} \lambda_0 & \lambda_1 \\ \mu_0 & \mu_1 \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial \xi_0} \\ \frac{\partial p}{\partial \xi_1} \end{bmatrix} \\ &= \sum_{|\alpha|=k-1} \frac{k}{A^{(1)}} \binom{|\alpha|}{\alpha} \begin{bmatrix} \lambda_0 & \lambda_1 \\ \mu_0 & \mu_1 \end{bmatrix} \begin{bmatrix} \triangle_{13} b_\alpha^{(1)} \\ \triangle_{23} b_\alpha^{(1)} \end{bmatrix} \xi_0^{\alpha_0} \xi_1^{\alpha_1} \xi_2^{\alpha_2}, \end{aligned} \tag{8}$$

where

$$\triangle_{13} b_\alpha^{(1)} := b^{(1)}_{\alpha_0+1\alpha_1\alpha_2} - b^{(1)}_{\alpha_0\alpha_1\alpha_2+1},$$

$$\triangle_{23} b_\alpha^{(1)} := b^{(1)}_{\alpha_0\alpha_1+1\alpha_2} - b^{(1)}_{\alpha_0\alpha_1\alpha_2+1},$$

and

$$\begin{bmatrix} \frac{\partial^2 p}{\partial x^2} \\ \frac{\partial^2 p}{\partial y^2} \\ \frac{\partial^2 p}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} (\frac{\partial \xi_0}{\partial x})^2 & (\frac{\partial \xi_1}{\partial x})^2 & 2\frac{\partial \xi_0}{\partial x}\frac{\partial \xi_1}{\partial x} \\ (\frac{\partial \xi_0}{\partial y})^2 & (\frac{\partial \xi_1}{\partial y})^2 & 2\frac{\partial \xi_0}{\partial y}\frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_0}{\partial x}\frac{\partial \xi_0}{\partial y} & \frac{\partial \xi_1}{\partial x}\frac{\partial \xi_1}{\partial y} & \frac{\partial \xi_0}{\partial x}\frac{\partial \xi_1}{\partial y} + \frac{\partial \xi_1}{\partial x}\frac{\partial \xi_0}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 p}{\partial {\xi_0}^2} \\ \frac{\partial^2 p}{\partial {\xi_1}^2} \\ \frac{\partial^2 p}{\partial \xi_0 \partial \xi_1} \end{bmatrix}$$

$$= \frac{1}{(A^{(1)})^2} \begin{bmatrix} (\lambda_0)^2 & (\lambda_1)^2 & 2\lambda_0\lambda_1 \\ (\mu_0)^2 & (\mu_1)^2 & 2\mu_0\mu_1 \\ \lambda_0\mu_0 & \lambda_1\mu_1 & \lambda_0\mu_1 + \lambda_1\mu_0 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 p}{\partial {\xi_0}^2} \\ \frac{\partial^2 p}{\partial {\xi_1}^2} \\ \frac{\partial^2 p}{\partial \xi_0 \partial \xi_1} \end{bmatrix}$$

$$= \sum_{|\alpha|=k-2} \frac{k(k-1)}{(A^{(1)})^2} \binom{|\alpha|}{\alpha} \begin{bmatrix} (\lambda_0)^2 & (\lambda_1)^2 & 2\lambda_0\lambda_1 \\ (\mu_0)^2 & (\mu_1)^2 & 2\mu_0\mu_1 \\ \lambda_0\mu_0 & \lambda_1\mu_1 & \lambda_0\mu_1 + \lambda_1\mu_0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} \triangle_{13}\, \triangle_{13}\, b_\alpha^{(1)} \\ \triangle_{23}\, \triangle_{23}\, b_\alpha^{(1)} \\ \triangle_{13}\, \triangle_{23}\, b_\alpha^{(1)} \end{bmatrix} \xi_0^{\alpha_0} \xi_1^{\alpha_1} \xi_2^{\alpha_2}. \tag{9}$$

Let $f \in C^2(\Omega)$. We consider the following interpolation problem on the simplex $\delta_1 = [v_0, v_1, v_2]$:

Find a quintic bivariate polynomial $p \in \pi_5$ such that

$$\begin{cases} p|_{v_i} = f|_{v_i} := f_i, & i = 0,1,2 \\ \frac{\partial p}{\partial x}|_{v_i} = \frac{\partial f}{\partial x}|_{v_i} := D_x f_i, & i = 0,1,2 \\ \frac{\partial p}{\partial y}|_{v_i} = \frac{\partial f}{\partial y}|_{v_i} := D_y f_i, & i = 0,1,2 \\ \frac{\partial^2 p}{\partial x^2}|_{v_i} = \frac{\partial^2 f}{\partial x^2}|_{v_i} := D_{xx} f_i, & i = 0,1,2 \\ \frac{\partial^2 p}{\partial y^2}|_{v_i} = \frac{\partial^2 f}{\partial y^2}|_{v_i} := D_{yy} f_i, & i = 0,1,2 \\ \frac{\partial^2 p}{\partial x \partial y}|_{v_i} = \frac{\partial^2 f}{\partial x \partial y}|_{v_i} := D_{xy} f_i, & i = 0,1,2 \end{cases}. \tag{10}$$

Substituting (7), (8) and (9) into (10), we obtain equivalent interpolation conditions in terms of the B-net ordinate $b_\alpha^{(1)}$ as follows:

$$f_0 = b^{(1)}_{500}, \quad f_1 = b^{(1)}_{050}, \quad f_2 = b^{(1)}_{005},$$

$$D_x f_0 = \frac{5}{A^{(1)}} \left[\lambda_0 b^{(1)}_{500} + \lambda_1 b^{(1)}_{410} + \lambda_2 b^{(1)}_{401} \right],$$

$$D_x f_1 = \frac{5}{A^{(1)}} \left[\lambda_1 b^{(1)}_{050} + \lambda_2 b^{(1)}_{041} + \lambda_0 b^{(1)}_{140} \right],$$

$$D_x f_2 = \frac{5}{A^{(1)}}\left[\lambda_2 b_{005}^{(1)} + \lambda_0 b_{104}^{(1)} + \lambda_1 b_{014}^{(1)}\right],$$
$$D_y f_0 = \frac{5}{A^{(1)}}\left[\mu_0 b_{500}^{(1)} + \mu_1 b_{410}^{(1)} + \mu_2 b_{401}^{(1)}\right],$$
$$D_y f_1 = \frac{5}{A^{(1)}}\left[\mu_1 b_{050}^{(1)} + \mu_2 b_{041}^{(1)} + \mu_0 b_{140}^{(1)}\right],$$
$$D_y f_2 = \frac{5}{A^{(1)}}\left[\mu_2 b_{005}^{(1)} + \mu_0 b_{104}^{(1)} + \mu_1 b_{014}^{(1)}\right],$$

$$D_{xx} f_0 = c\Big[\lambda_0^2 b_{500}^{(1)} + \lambda_1^2 b_{320}^{(1)} + \lambda_2^2 b_{302}^{(1)} + 2\lambda_0\lambda_2 b_{401}^{(1)} + 2\lambda_1\lambda_0 b_{410}^{(1)} + \\ + 2\lambda_2\lambda_1 b_{311}^{(1)}\Big],$$
$$D_{xx} f_1 = c\Big[\lambda_1^2 b_{050}^{(1)} + \lambda_2^2 b_{032}^{(1)} + \lambda_0^2 b_{230}^{(1)} + 2\lambda_1\lambda_0 b_{140}^{(1)} + 2\lambda_2\lambda_1 b_{041}^{(1)} + \\ + 2\lambda_0\lambda_2 b_{131}^{(1)}\Big],$$
$$D_{xx} f_2 = c\Big[\lambda_2^2 b_{005}^{(1)} + \lambda_0^2 b_{203}^{(1)} + \lambda_1^2 b_{023}^{(1)} + 2\lambda_2\lambda_1 b_{014}^{(1)} + 2\lambda_0\lambda_2 b_{104}^{(1)} + \\ + 2\lambda_1\lambda_0 b_{113}^{(1)}\Big],$$
$$D_{yy} f_0 = c\Big[\mu_0^2 b_{500}^{(1)} + \mu_1^2 b_{320}^{(1)} + \mu_2^2 b_{302}^{(1)} + 2\mu_0\mu_2 b_{401}^{(1)} + 2\mu_1\mu_0 b_{410}^{(1)} + \\ + 2\mu_2\mu_1 b_{311}^{(1)}\Big],$$
$$D_{yy} f_1 = c\Big[\mu_1^2 b_{050}^{(1)} + \mu_2^2 b_{032}^{(1)} + \mu_0^2 b_{230}^{(1)} + 2\mu_1\mu_0 b_{140}^{(1)} + 2\mu_2\mu_1 b_{041}^{(1)} + \\ + 2\mu_0\mu_2 b_{131}^{(1)}\Big],$$
$$D_{yy} f_2 = c\Big[\mu_2^2 b_{005}^{(1)} + \mu_0^2 b_{203}^{(1)} + \mu_1^2 b_{023}^{(1)} + 2\mu_2\mu_1 b_{014}^{(1)} + 2\mu_0\mu_2 b_{104}^{(1)} + \\ + 2\mu_1\mu_0 b_{113}^{(1)}\Big],$$
$$D_{xy} f_0 = c\Big[\lambda_0\mu_0 b_{500}^{(1)} + \lambda_1\mu_1 b_{320}^{(1)} + \lambda_2\mu_2 b_{302}^{(1)} + (\lambda_0\mu_2 + \lambda_2\mu_0) b_{401}^{(1)} + \\ + (\lambda_1\mu_0 + \lambda_0\mu_1) b_{410}^{(1)} + (\lambda_2\mu_1 + \lambda_1\mu_2) b_{311}^{(1)}\Big],$$
$$D_{xy} f_1 = c\Big[\lambda_1\mu_1 b_{050}^{(1)} + \lambda_2\mu_2 b_{032}^{(1)} + \lambda_0\mu_0 b_{230}^{(1)} + (\lambda_1\mu_0 + \lambda_0\mu_1) b_{140}^{(1)} + \\ + (\lambda_2\mu_1 + \lambda_1\mu_2) b_{041}^{(1)} + (\lambda_0\mu_2 + \lambda_2\mu_0) b_{131}^{(1)}\Big],$$
$$D_{xy} f_2 = c\Big[\lambda_2\mu_2 b_{005}^{(1)} + \lambda_0\mu_0 b_{203}^{(1)} + \lambda_1\mu_1 b_{023}^{(1)} + (\lambda_2\mu_1 + \lambda_1\mu_2) b_{014}^{(1)} + \\ + (\lambda_0\mu_2 + \lambda_2\mu_0) b_{104}^{(1)} + (\lambda_1\mu_0 + \lambda_0\mu_1) b_{113}^{(1)}\Big]$$

with $c = 20/(A^{(1)})^2$. Then we can solve for some B-net ordinates $b_\alpha^{(1)}$ of p as follows:

$$
\begin{aligned}
b_{500}^{(1)} &= f_0, \quad b_{050}^{(1)} = f_1, \quad b_{005}^{(1)} = f_2,\\
b_{410}^{(1)} &= f_0 + \frac{1}{5}\left[\mu_2 D_x f_0 - \lambda_2 D_y f_0\right],\\
b_{401}^{(1)} &= f_0 - \frac{1}{5}\left[\mu_1 D_x f_0 - \lambda_1 D_y f_0\right],\\
b_{041}^{(1)} &= f_1 + \frac{1}{5}\left[\mu_0 D_x f_1 - \lambda_0 D_y f_1\right],\\
b_{140}^{(1)} &= f_1 - \frac{1}{5}\left[\mu_2 D_x f_1 - \lambda_2 D_y f_1\right],\\
b_{104}^{(1)} &= f_2 + \frac{1}{5}\left[\mu_1 D_x f_2 - \lambda_1 D_y f_2\right],\\
b_{014}^{(1)} &= f_2 - \frac{1}{5}\left[\mu_0 D_x f_2 - \lambda_0 D_y f_2\right],\\
b_{302}^{(1)} &= -b_{500}^{(1)} + 2b_{401}^{(1)} + \frac{1}{20}\left[\mu_1^2 D_{xx} f_0 - 2\lambda_1\mu_1 D_{xy} f_0 + \lambda_1^2 D_{yy} f_0\right],\\
b_{320}^{(1)} &= -b_{500}^{(1)} + 2b_{410}^{(1)} + \frac{1}{20}\left[\mu_2^2 D_{xx} f_0 - 2\lambda_2\mu_2 D_{xy} f_0 + \lambda_2^2 D_{yy} f_0\right],\\
b_{311}^{(1)} &= -b_{500}^{(1)} + b_{410}^{(1)} + b_{401}^{(1)}\\
&\quad - \frac{1}{20}\left[\mu_1\mu_2 D_{xx} f_0 - (\lambda_2\mu_1 + \lambda_1\mu_2) D_{xy} f_0 + \lambda_1\lambda_2 D_{yy} f_0\right],\\
b_{230}^{(1)} &= -b_{050}^{(1)} + 2b_{140}^{(1)} + \frac{1}{20}\left[\mu_2^2 D_{xx} f_1 - 2\lambda_2\mu_2 D_{xy} f_1 + \lambda_2^2 D_{yy} f_1\right],\\
b_{032}^{(1)} &= -b_{050}^{(1)} + 2b_{041}^{(1)} + \frac{1}{20}\left[\mu_0^2 D_{xx} f_1 - 2\lambda_0\mu_0 D_{xy} f_1 + \lambda_0^2 D_{yy} f_1\right],\\
b_{131}^{(1)} &= -b_{050}^{(1)} + b_{041}^{(1)} + b_{140}^{(1)}\\
&\quad - \frac{1}{20}\left[\mu_2\mu_0 D_{xx} f_1 - (\lambda_0\mu_2 + \lambda_2\mu_0) D_{xy} f_1 + \lambda_2\lambda_0 D_{yy} f_1\right],\\
b_{023}^{(1)} &= -b_{005}^{(1)} + 2b_{014}^{(1)} + \frac{1}{20}\left[\mu_0^2 D_{xx} f_2 - 2\lambda_0\mu_0 D_{xy} f_2 + \lambda_0^2 D_{yy} f_2\right],\\
b_{203}^{(1)} &= -b_{005}^{(1)} + 2b_{104}^{(1)} + \frac{1}{20}\left[\mu_1^2 D_{xx} f_2 - 2\lambda_1\mu_1 D_{xy} f_2 + \lambda_1^2 D_{yy} f_2\right],\\
b_{113}^{(1)} &= -b_{005}^{(1)} + b_{104}^{(1)} + b_{014}^{(1)}\\
&\quad - \frac{1}{20}\left[\mu_0\mu_1 D_{xx} f_2 - (\lambda_1\mu_0 + \lambda_0\mu_1) D_{xy} f_2 + \lambda_0\lambda_1 D_{yy} f_2\right].
\end{aligned} \tag{11}
$$

We note that there are three B-net ordinates $b_{221}^{(1)}, b_{212}^{(1)}$ and $b_{122}^{(1)}$ of p that are not given. In fact, they can not be uniquely determined by the interpolation condition (9). It is easy to show that, if we add three additional interpolation

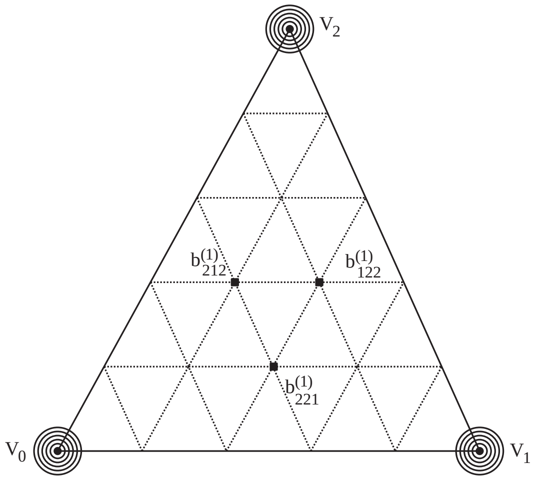

Fig. 1. The free parameters of $p \in \Pi_5$ on a triangle

conditions as

$$p|_{v_i} = f|_{v_i} := f_i, \quad i = 3, 4, 5,$$

where v_3, v_4 and v_5 are internal points of δ_1 with barycentric coordinates $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$, $(\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$ and $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ respectively, then $b^{(1)}_{221}$, $b^{(1)}_{212}$ and $b^{(1)}_{122}$ are uniquely determined. However, we are not going to do so. For convenience we will keep them undetermined at first so that the free parameters to any polynomial p on a simplex can be chosen as the function values and all the first and second derivative values at the three vertices as well as the three B-net ordinates $b^{(1)}_{221}$, $b^{(1)}_{212}$ and $b^{(1)}_{122}$ (see Fig. 1). In the following, we will see that this group of new free parameters of quintic polynomials can be used to simplify the smoothness conditions and conformality conditions of bivariate C^2 quintic splines.

As shown in Fig. 2, let $\delta_1 = [v_0, v_1, v_2]$ and $\delta_2 = [v_0, v_1, v_3]$ be two adjacent triangles with a common edge $[v_0, v_1]$ where $v_i = (x_i, y_i), i = 0, 1, 2, 3$. For convenience, let $A^{(1)}$ denote the area of the triangle δ_1 and

$$A^{(1)}_0 := \text{area}[v_3, v_1, v_2],$$
$$A^{(1)}_1 := \text{area}[v_1, v_3, v_2],$$
$$A^{(1)}_2 := \text{area}[v_0, v_1, v_3].$$

Clear $A^{(1)}_2 = A^{(2)}$. Suppose that the piecewise quintic polynomial function $F(x, y)$ is defined on $\delta_1 \cup \delta_2$ by

$$F|_{\delta_1} := f = \sum_{|\alpha|=5} b^{(1)}_\alpha B_{\alpha,\delta_1},$$

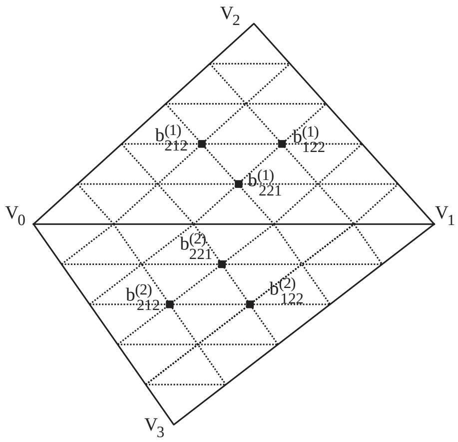

Fig. 2. B-net of a quintic spline on $\delta_1 \cup \delta_2$

$$F|_{\delta_2} := g = \sum_{|\alpha|=5} b_\alpha^{(2)} B_{\alpha,\delta_2}.$$

Let $g_i = g(v_i)$, $i = 0, 1, 2$, $D_x g_i := \frac{\partial g}{\partial x}|_{v_i}$, $D_y g_i := \frac{\partial g}{\partial y}|_{v_i}$, $D_{xx} g_i := \frac{\partial^2 g}{\partial x^2}|_{v_i}$, $D_{yy} g_i := \frac{\partial^2 g}{\partial y^2}|_{v_i}$, $D_{xy} g_i := \frac{\partial^2 g}{\partial x \partial y}|_{v_i}$, $i = 0, 1$. Then the smoothness conditions of the bivariate C^2 quintic splines in Theorem 1 can be simplified as follows:

$$b_\alpha^{(2)} = b_\alpha^{(1)}, \qquad \alpha = (\alpha_0, \alpha_1, 0), |\alpha| = 5,$$

$$b_{\alpha+\mathbf{e}^3}^{(2)} = \sum_{|\beta|=1} b_{\alpha+\beta}^{(1)} \frac{(A_0^{(1)})^{\beta_0} (A_1^{(1)})^{\beta_1} (A_2^{(1)})^{\beta_2}}{A^{(1)}}, \qquad \alpha = (\alpha_0, \alpha_1, 0), |\alpha| = 4,$$

$$b_{\alpha+2\mathbf{e}^3}^{(2)} = \sum_{|\beta|=2} \binom{|\beta|}{\beta} b_{\alpha+\beta}^{(1)} \frac{(A_0^{(1)})^{\beta_0} (A_1^{(1)})^{\beta_1} (A_2^{(1)})^{\beta_2}}{(A^{(1)})^2}, \quad \alpha = (\alpha_0, \alpha_1, 0), |\alpha| = 3.$$

More specifically, we have

$$b_{500}^{(2)} = b_{500}^{(1)}, \tag{12}$$
$$b_{410}^{(2)} = b_{410}^{(1)}, \tag{13}$$
$$b_{320}^{(2)} = b_{320}^{(1)}, \tag{14}$$
$$b_{230}^{(2)} = b_{230}^{(1)}, \tag{15}$$
$$b_{140}^{(2)} = b_{140}^{(1)}, \tag{16}$$
$$b_{050}^{(2)} = b_{050}^{(1)}, \tag{17}$$
$$b_{401}^{(2)} = \frac{1}{A^{(1)}} \left[b_{500}^{(1)} A_0^{(1)} + b_{410}^{(1)} A_1^{(1)} + b_{401}^{(1)} A_2^{(1)} \right], \tag{18}$$

$$b_{311}^{(2)} = \frac{1}{A^{(1)}}\left[b_{410}^{(1)}A_0^{(1)} + b_{320}^{(1)}A_1^{(1)} + b_{311}^{(1)}A_2^{(1)}\right], \tag{19}$$

$$b_{221}^{(2)} = \frac{1}{A^{(1)}}\left[b_{320}^{(1)}A_0^{(1)} + b_{230}^{(1)}A_1^{(1)} + b_{221}^{(1)}A_2^{(1)}\right], \tag{20}$$

$$b_{131}^{(2)} = \frac{1}{A^{(1)}}\left[b_{230}^{(1)}A_0^{(1)} + b_{140}^{(1)}A_1^{(1)} + b_{131}^{(1)}A_2^{(1)}\right], \tag{21}$$

$$b_{041}^{(2)} = \frac{1}{A^{(1)}}\left[b_{140}^{(1)}A_0^{(1)} + b_{050}^{(1)}A_1^{(1)} + b_{041}^{(1)}A_2^{(1)}\right], \tag{22}$$

$$\begin{aligned} b_{302}^{(2)} = \frac{1}{(A^{(1)})^2}\Big[&b_{500}^{(1)}(A_0^{(1)})^2 + b_{320}^{(1)}(A_1^{(1)})^2 + b_{302}^{(1)}(A_2^{(1)})^2 + \\ &+2b_{410}^{(1)}A_0^{(1)}A_1^{(1)} + 2b_{401}^{(1)}A_0^{(1)}A_2^{(1)} + 2b_{311}^{(1)}A_1^{(1)}A_2^{(1)}\Big], \end{aligned} \tag{23}$$

$$\begin{aligned} b_{212}^{(2)} = \frac{1}{(A^{(1)})^2}\Big[&b_{410}^{(1)}(A_0^{(1)})^2 + b_{230}^{(1)}(A_1^{(1)})^2 + b_{212}^{(1)}(A_2^{(1)})^2 + \\ &+2b_{320}^{(1)}A_0^{(1)}A_1^{(1)} + 2b_{311}^{(1)}A_0^{(1)}A_2^{(1)} + 2b_{221}^{(1)}A_1^{(1)}A_2^{(1)}\Big], \end{aligned} \tag{24}$$

$$\begin{aligned} b_{122}^{(2)} = \frac{1}{(A^{(1)})^2}\Big[&b_{140}^{(1)}(A_1^{(1)})^2 + b_{122}^{(1)}(A_2^{(1)})^2 + b_{320}^{(1)}(A_0^{(1)})^2 + \\ &+2b_{230}^{(1)}A_0^{(1)}A_1^{(1)} + 2b_{221}^{(1)}A_0^{(1)}A_2^{(1)} + 2b_{131}^{(1)}A_1^{(1)}A_2^{(1)}\Big], \end{aligned} \tag{25}$$

$$\begin{aligned} b_{032}^{(2)} = \frac{1}{(A^{(1)})^2}\Big[&b_{050}^{(1)}(A_1^{(1)})^2 + b_{032}^{(1)}(A_2^{(1)})^2 + b_{230}^{(1)}(A_0^{(1)})^2 + \\ &+2b_{140}^{(1)}A_0^{(1)}A_1^{(1)} + 2b_{041}^{(1)}A_1^{(1)}A_2^{(1)} + 2b_{131}^{(1)}A_0^{(1)}A_2^{(1)}\Big]. \end{aligned} \tag{26}$$

Corresponding to the splines determined by the given interpolation conditions, we have the following.

Theorem 3 *Suppose that the piecewise quintic polynomial function $F(x, y)$ is defined on $\delta_1 \cup \delta_2$ and $F|_{\delta_1} = f$, $F|_{\delta_2} = g$, where f and g are determined by the interpolation conditions (10). Then $F(x, y) \in C^1(\delta_1 \cup \delta_2)$ if and only if*

$$b_{221}^{(2)} = \frac{1}{A^{(1)}}\left[b_{320}^{(1)}A_0^{(1)} + b_{230}^{(1)}A_1^{(1)} + b_{221}^{(1)}A_2^{(1)}\right], \tag{27}$$

where $b_{320}^{(1)}$ and $b_{230}^{(1)}$ are defined as in (12).

Theorem 4 *Suppose that the piecewise quintic polynomial function $F(x, y)$ is defined on $\delta_1 \cup \delta_2$ and $F|_{\delta_1} = f$, $F|_{\delta_2} = g$, where f and g are determined by the interpolation conditions (10). Then $F(x, y) \in C^2(\delta_1 \cup \delta_2)$ if and only if*

$$b_{221}^{(2)} = \frac{1}{A^{(1)}}\left[b_{320}^{(1)}A_0^{(1)} + b_{230}^{(1)}A_1^{(1)} + b_{221}^{(1)}A_2^{(1)}\right],$$

$$b_{212}^{(2)} = \frac{1}{(A^{(1)})^2}\Big[b_{410}^{(1)}(A_0^{(1)})^2 + b_{230}^{(1)}(A_1^{(1)})^2 + b_{212}^{(1)}(A_2^{(1)})^2 + \\ +2b_{320}^{(1)}A_0^{(1)}A_1^{(1)} + 2b_{311}^{(1)}A_0^{(1)}A_2^{(1)} + 2b_{221}^{(1)}A_1^{(1)}A_2^{(1)}\Big],$$

$$b_{122}^{(2)} = \frac{1}{(A^{(1)})^2}\Big[b_{140}^{(1)}(A_1^{(1)})^2 + b_{122}^{(1)}(A_2^{(1)})^2 + b_{320}^{(1)}(A_0^{(1)})^2 + \\ +2b_{230}^{(1)}A_0^{(1)}A_1^{(1)} + 2b_{221}^{(1)}A_0^{(1)}A_2^{(1)} + 2b_{131}^{(1)}A_1^{(1)}A_2^{(1)}\Big],$$

where $b_{320}^{(1)}, b_{230}^{(1)}, b_{410}^{(1)}, b_{140}^{(1)}, b_{311}^{(1)}$ and $b_{131}^{(1)}$ are defined as in (12).

Proof We prove only Theorem 4. First, we notice that equalities (12) and (17) are respectively equivalent to

$$g_0 = f_0, \quad g_1 = f_1. \tag{28}$$

Next, if we set

$$\begin{array}{lll} \lambda_0' := y_1 - y_3, & \lambda_1' := y_0 - y_3, & \lambda_2' := y_1 - y_2 = \lambda_2, \\ \mu_0' := -(x_1 - x_3), & \mu_1' := -(x_0 - x_3), & \mu_2' := -(x_1 - x_2) = \mu_2, \end{array}$$

then, from

$$\begin{vmatrix} -\mu_2 & \lambda_2 & \lambda_2 \\ \mu_0 & -\lambda_0 & -\lambda_0 \\ \mu_0' & -\lambda_0' & -\lambda_0' \end{vmatrix} = 0,$$

we have

$$\lambda_2 \begin{vmatrix} \mu_0 & -\lambda_0 \\ \mu_0' & -\lambda_0' \end{vmatrix} + \lambda_0 \begin{vmatrix} -\mu_2 & \lambda_2 \\ \mu_0' & -\lambda_0' \end{vmatrix} - \lambda_0' \begin{vmatrix} -\mu_2 & \lambda_2 \\ \mu_0 & -\lambda_0 \end{vmatrix} = 0,$$

i.e.,

$$\lambda_2 A_0^{(1)} + \lambda_0' A^{(1)} = \lambda_0 A_2^{(1)}. \tag{29}$$

Similarly,

$$\lambda_2 A_1^{(1)} + \lambda_1' A^{(1)} = \lambda_1 A_2^{(1)}. \tag{30}$$

So from (13) and (18), we have

$$\begin{aligned} D_x g_0 &= \frac{5}{A^{(2)}}(\lambda_0' b_{500}^{(2)} + \lambda_1' b_{410}^{(2)} + \lambda_2' b_{401}^{(2)}) \\ &= \frac{5}{A_2^{(1)}}(\lambda_0' b_{500}^{(1)} + \lambda_1' b_{410}^{(1)} + \lambda_2 b_{401}^{(2)}) \\ &= \frac{5}{A_2^{(1)}}\left[\lambda_0' b_{500}^{(1)} + \lambda_1' b_{410}^{(1)} + \lambda_2 \frac{1}{A^{(1)}}(b_{500}^{(1)} A_0^{(1)} + b_{410}^{(1)} A_1^{(1)} + b_{401}^{(1)} A_2^{(1)})\right] \end{aligned}$$

$$= \frac{5}{A^{(1)}A_2^{(1)}}\Big[(\lambda_2 A_0^{(1)} + \lambda_0' A^{(1)})b_{500}^{(1)} + (\lambda_2 A_1^{(1)} + \lambda_1' A^{(1)})b_{410}^{(1)} + \lambda_2 A_2^{(1)} b_{401}^{(1)}\Big]$$

$$= \frac{5}{A^{(1)}}(\lambda_0 b_{500}^{(1)} + \lambda_1 b_{410}^{(1)} + \lambda_2 b_{401}^{(1)}] = D_x f_0. \tag{31}$$

Furthermore, note that (13) is equivalent to

$$g_0 + \frac{1}{5}\left[\mu_2' D_x g_0 - \lambda_2' D_y g_0\right] = f_0 + \frac{1}{5}\left[\mu_2 D_x f_0 - \lambda_2 D_y f_0\right],$$

so that, together with (28) and (31), we obtain

$$D_y g_0 = D_y f_0. \tag{32}$$

Similarly, (16) and (22) are equivalent to

$$D_x g_1 = D_x f_1, \quad D_y g_1 = D_y f_1. \tag{33}$$

Next we examine the second derivatives. Combining (14), (19) and (23), we have

$$\begin{aligned}
D_{xx} g_0 = {} & \frac{20}{(A^{(2)})^2}\Big[(\lambda_0')^2 b_{500}^{(2)} + (\lambda_1')^2 b_{320}^{(2)} + (\lambda_2')^2 b_{302}^{(2)} + \\
& \quad + 2\lambda_0'\lambda_2' b_{401}^{(2)} + 2\lambda_1'\lambda_0' b_{410}^{(2)} + 2\lambda_2'\lambda_1' b_{311}^{(2)}\Big] \\
= {} & \frac{20}{(A^{(2)})^2}\Big\{(\lambda_0')^2 b_{500}^{(1)} + (\lambda_1')^2 b_{320}^{(1)} + 2\lambda_1'\lambda_0' b_{410}^{(1)} + \\
& \quad + \lambda_2^2 \frac{1}{(A^{(1)})^2}\Big[b_{500}^{(1)}(A_0^{(1)})^2 + b_{320}^{(1)}(A_1^{(1)})^2 + b_{302}^{(1)}(A_2^{(1)})^2 + \\
& \qquad + 2b_{410}^{(1)}A_0^{(1)}A_1^{(1)} + 2b_{401}^{(1)}A_0^{(1)}A_2^{(1)} + 2b_{311}^{(1)}A_1^{(1)}A_2^{(1)}\Big] + \\
& \quad + 2\lambda_0'\lambda_2 \frac{1}{A^{(1)}}\Big[b_{500}^{(1)}A_0^{(1)} + b_{410}^{(1)}A_1^{(1)} + b_{401}^{(1)}A_2^{(1)}\Big] + \\
& \quad + 2\lambda_2\lambda_1' \frac{1}{A^{(1)}}\Big[b_{410}^{(1)}A_0^{(1)} + b_{320}^{(1)}A_1^{(1)} + b_{311}^{(1)}A_2^{(1)}\Big]\Big\} \\
= {} & \frac{20}{(A^{(2)})^2}\Big\{\frac{1}{(A^{(1)})^2}(\lambda_2 A_0^{(1)} + \lambda_0' A^{(1)})^2 b_{500}^{(1)} + \\
& \quad + \frac{1}{(A^{(1)})^2}(\lambda_2 A_1^{(1)} + \lambda_1' A^{(1)})^2 b_{320}^{(1)} + \\
& \quad + \lambda_2^2 \frac{(A_2^{(1)})^2}{(A^{(1)})^2} b_{302}^{(1)} + 2\lambda_2(\lambda_2 A_0^{(1)} + \lambda_0' A^{(1)})\frac{A_2^{(1)}}{(A^{(1)})^2} b_{401}^{(1)} + \\
& \quad + 2(\lambda_2 A_0^{(1)} + \lambda_0' A^{(1)})(\lambda_2 A_1^{(1)} + \lambda_1' A^{(1)})\frac{1}{(A^{(1)})^2} b_{410}^{(1)} +
\end{aligned}$$

$$
\left. +2\lambda_2(\lambda_2 A_1^{(1)} + \lambda_1' A^{(1)}) \frac{A_2^{(1)}}{(A^{(1)})^2} b_{311}^{(1)} \right\}
$$

$$
\begin{aligned}
&= \frac{20}{(A^{(1)})^2} \Big[\lambda_0^2 b_{500}^{(1)} + \lambda_1^2 b_{320}^{(1)} + \lambda_2^2 b_{302}^{(1)} + \\
&\qquad + 2\lambda_0\lambda_2 b_{401}^{(1)} + 2\lambda_1\lambda_0 b_{410}^{(1)} + 2\lambda_2\lambda_1 b_{311}^{(1)} \Big] \\
&= D_{xx} f_0. \qquad (34)
\end{aligned}
$$

In addition, from (14), (19) and (23), we obtain

$$
D_{yy} g_0 = D_{yy} f_0, \quad D_{xy} g_0 = D_{xy} f_0. \qquad (35)
$$

And similarly, we can prove that (15), (21) and (26) are equivalent to

$$
D_{xx} g_1 = D_{xx} f_1, \quad D_{yy} g_1 = D_{yy} f_1, \quad D_{xy} g_1 = D_{xy} f_1. \qquad (36)
$$

This completes the proof of the theorem. □

Theorem 4 indicates that the quintic spline $F(x, y) \in S_5^2(\delta_1 \cup \delta_2)$ can be determined by the given values $F|_{v_i}$, $D_x F|_{v_i}$, $D_y F|_{v_i}$, $D_{xx} F|_{v_i}$, $D_{yy} F|_{v_i}$, $D_{xy} F|_{v_i}$, $i = 0, 1, 2, 3$ and the three B-net ordinates $b_{221}^{(1)}$, $b_{212}^{(1)}$ and $b_{122}^{(1)}$ (or $b_{221}^{(2)}$, $b_{212}^{(2)}$ and $b_{122}^{(2)}$). Thus there are twenty seven free parameters in total to uniquely determine a quintic C^2 spline on $\delta_1 \cup \delta_2$.

Similarly, for any quartic polynomial $p(x, y)$ on a simplex, the free parameters of p can be chosen as the function values and all the first derivative values at the three vertices and the function values on all the middle points of three edges and the three B-net ordinates $b_{211}^{(1)}$, $b_{112}^{(1)}$ and $b_{121}^{(1)}$, which are plotted in Fig. 3.

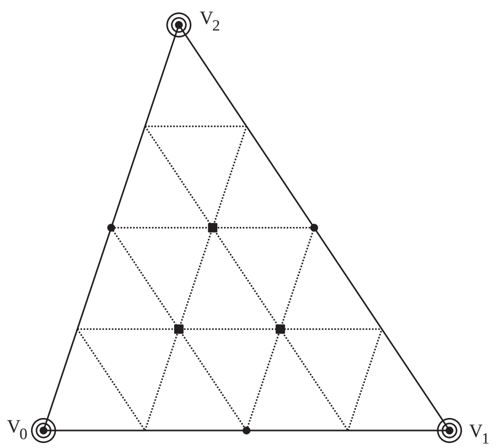

Fig. 3. The free parameters of $p \in \Pi_4$ on a triangle

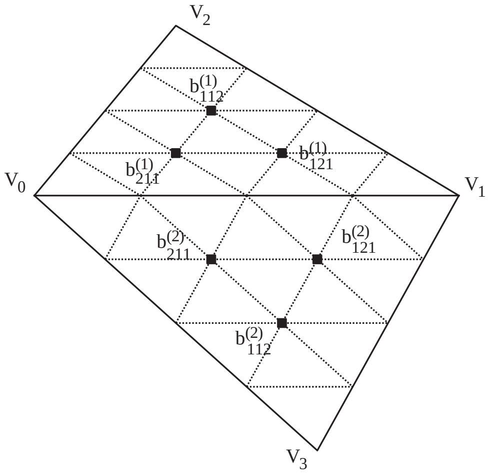

Fig. 4. B-net of a quartic spline on $\delta_1 \cup \delta_2$

In addition, we have the following smoothness conditions for C^1 quartic splines.

Theorem 5 *Suppose that the piecewise quartic polynomial function $F(x, y)$ is defined on $\delta_1 \cup \delta_2$ as shown in Fig. 4 and that $F|_{\delta_1} = f$, $F|_{\delta_2} = g$ where f and g satisfy the following conditions*

$$\begin{cases} f_i &= g_i, \quad i = 0, 1 \\ f(\frac{v_0+v_1}{2}) &= g(\frac{v_0+v_1}{2}) \\ D_x f_i &= D_x g_i, i = 0, 1 \\ D_y f_i &= D_y g_i, i = 0, 1. \end{cases} \tag{37}$$

Then $F(x, y) \in C^1(\delta_1 \cup \delta_2)$ if and only if

$$b_{211}^{(2)} = \frac{1}{A^{(1)}}\left[b_{310}^{(1)} A_0^{(1)} + b_{220}^{(1)} A_1^{(1)} + b_{211}^{(1)} A_2^{(1)}\right],$$
$$b_{121}^{(2)} = \frac{1}{A^{(1)}}\left[b_{220}^{(1)} A_0^{(1)} + b_{130}^{(1)} A_1^{(1)} + b_{121}^{(1)} A_2^{(1)}\right],$$

where

$$b_{310}^{(1)} = f_0 + \frac{1}{4}(\mu_2 D_x f_0 - \lambda_2 D_y f_0),$$
$$b_{130}^{(1)} = f_1 - \frac{1}{4}(\mu_2 D_x f_1 - \lambda_2 D_y f_1),$$
$$b_{220}^{(1)} = \frac{1}{6}(16 f(\frac{v_0 + v_1}{2}) - 5 f_0 - 5 f_1 - \mu_2 D_x f_0 + \lambda_2 D_y f_0 + \\ + \mu_2 D_x f_1 - \lambda_2 D_y f_1).$$

We can see from Theorem 5 that the quartic spline $F(x, y) \in S_4^1(\delta_1 \cup \delta_2)$ can be determined by the given values $F|_{v_i}$, $D_x F|_{v_i}$, $D_y F|_{v_i}$, $i = 0, 1, 2, 3$, $F(\frac{v_0+v_1}{2})$, $F(\frac{v_0+v_2}{2})$, $F(\frac{v_0+v_3}{2})$, $F(\frac{v_1+v_2}{2})$, $F(\frac{v_1+v_3}{2})$, $b_{112}^{(1)}$, $b_{112}^{(1)}$, and $b_{211}^{(1)}$ and $b_{121}^{(1)}$ (or $b_{211}^{(2)}$ and $b_{121}^{(2)}$). Thus there are twenty one free parameters in total to uniquely determine a quartic C^1 spline on $\delta_1 \cup \delta_2$.

4 Conformality conditions of bivariate quartic splines

In this section, we discuss conformality conditions of bivariate quartic splines. Following the notation in [16], the union of all the triangles with the common vertex v of a triangulation Δ is called a *standard cell with interior vertex* v and denoted by Δ_v. The boundary vertices of Δ_v, in the counter-clockwise direction, are denoted by v_j, $j = 1, 2, \ldots, d$. The number of edges emanating from v is called the degree of v and denoted by $\deg(v)$. We call a triangulation $\triangle$ an odd- (even-) triangulation if the degree of each interior vertex in $\triangle$ is an odd (even) number. For a standard cell Δ_v with interior vertex v as shown in Fig. 5, we define

$$\begin{aligned} v_0 &= v, \quad e_j = [v_0, v_j], \\ A^{(j)} &= \text{area}[v_{j+1}, v_0, v_j,], \\ A_0^{(j)} &= \text{area}[v_{j+2}, v_0, v_j], \\ A_1^{(j)} &= \text{area}[v_{j+1}, v_{j+2}, v_j], \\ A_2^{(j)} &= \text{area}[v_{j+1}, v_0, v_{j+2}], \end{aligned}$$

where $j = 1, \ldots, d$, and $j + 1$ and $j + 2$ are taken $\text{mod}(d)$.

For a triangulation Δ, suppose that Δ_v is a standard cell with an interior vertex v of the triangulation Δ. Then the conditions (or linear equations) which a spline $s \in S_k^r(\Delta)$ satisfies around the vertex v are called conformality conditions. The conformality conditions in terms of smooth cofactors was first studied by Wang in [17]. In [18,19] (see also [20]), in order to give an integral representation of bivariate splines, Liu introduced so-called integral conformality conditions of bivariate splines. In [15], a simple conformality condition for bivariate cubic C^1 splines was given. Here, we give other conformality conditions on bivariate quartic super splines in terms of the new set of smoothness conditions obtained in Theorem 5.

The concept of general super splines was introduced by Schumaker in [21]. The subspace of super splines of smoothness r and degree $\leq k$ with enhanced smoothness order $\theta \geq r$ is defined as

$$S_k^{r,\theta}(\Delta) = \{s \in S_k^r(\Delta) : \ s \in C^\theta \text{ at each vertex of } \Delta\}.$$

We consider the conformality conditions for bivariate quartic splines based on the quartic super spline space $S_4^{0,1}(\Delta)$. For this purpose, corresponding

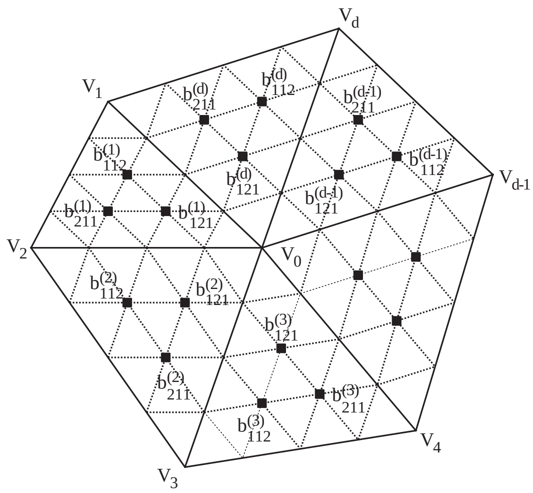

Fig. 5. The standard cell ∇_{v_0}

to Δ_v for an interior vertex $v := v_0 = (x_0, y_0)$ with $d = \deg(v)$ and vertices $v_j = (x_j, y_j),\ \ j = 1, \ldots, d$, being in the counter-clockwise direction, we define

$$s_j = s(v_j),\ \ D_x s_j = \frac{\partial s}{\partial x}|_{v_j},\ \ D_y s_j = \frac{\partial s}{\partial y}|_{v_j}$$

for $s \in S_4^{0,1}(\Delta_v)$ and $j = 0, \ldots, d$.

We have the following result on conformality conditions of bivariate quartic splines.

Theorem 6 *Suppose $s(x, y) \in S_4^{0,1}(\Delta_v)$ is a bivariate super quartic spline defined on a standard cell Δ_v with an interior vertex v. Then the conformality condition for $s(x, y) \in S_4^1(\Delta_v)$ is given by:*

i) if d is an even number ($d = 2N$), then

$$\sum_{j=1}^{2N} (-1)^j \frac{1}{A^{(j)} A^{(j+1)}} \left[\, b_{220}^{(j)} A_0^{(j)} + b_{130}^{(j)} A_1^{(j)} \,\right] = 0; \tag{38}$$

ii) if d is an odd number (d=2N+1), then

$$b_{121}^{(1)} = \frac{1}{2} \sum_{j=1}^{2N+1} (-1)^{j+1} \frac{A^{(1)}}{A^{(j)} A^{(j+1)}} \left[\, b_{220}^{(j)} A_0^{(j)} + b_{130}^{(j)} A_1^{(j)} \,\right]. \tag{39}$$

Where

$$b_{130}^{(j)} = s_0 + \frac{1}{4}(x_{j+1} - x_0)\, D_x s_0 + \frac{1}{4}(y_{j+1} - y_0)\, D_y s_0,$$

$$b_{220}^{(j)} = \frac{1}{6}\Big[16 s(\frac{v_0 + v_{j+1}}{2}) - 5s_0 - 5s_{j+1} + \\ +(x_{j+1} - x_0)(D_x s_0 - D_x s_{j+1}) + (y_{j+1} - y_0)(D_y s_0 - D_y s_{j+1})\Big].$$

Proof From Theorem 5, we have that, if $s \in S_4^{0,1}(\triangle_v)$, then $s \in S_4^1(\triangle_v)$ iff

$$b_{121}^{(j+1)} = [b_{220}^{(j)} A_0^{(j)} + b_{130}^{(j)} A_1^{(j)} + b_{121}^{(j)} A_2^{(j)}]/A^{(j)}$$

for $j = 1, \ldots, d$. In other words, we have

$$\begin{aligned} b_{121}^{(2)} &= [b_{220}^{(1)} A_0^{(1)} + b_{130}^{(1)} A_1^{(1)} + b_{121}^{(1)} A_2^{(1)}]/A^{(1)}, \\ b_{121}^{(3)} &= [b_{220}^{(2)} A_0^{(2)} + b_{130}^{(2)} A_1^{(2)} + b_{121}^{(2)} A_2^{(2)}]/A^{(2)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{121}^{(d)} &= [b_{220}^{(d-1)} A_0^{(d-1)} + b_{130}^{(d-1)} A_1^{(d-1)} + b_{121}^{(d-1)} A_2^{(d-1)}]/A^{(d-1)}, \\ b_{121}^{(1)} &= [b_{220}^{(d)} A_0^{(d)} + b_{130}^{(d)} A_1^{(d)} + b_{121}^{(d)} A_2^{(d)}]/A^{(d)}. \end{aligned}$$

Combining these equations and noticing that $A_2^{(j)} = -A^{(j+1)}$, we have

$$\begin{aligned} b_{121}^{(1)} &= \sum_{j=1}^{d} \Big(\prod_{\ell=j}^{d} \frac{A_2^{(\ell)}}{A^{(\ell)}}\Big) \frac{1}{A_2^{(j)}} [b_{220}^{(j)} A_0^{(j)} + b_{130}^{(j)} A_1^{(j)}] + \Big(\prod_{\ell=1}^{d} \frac{A_2^{(\ell)}}{A^{(\ell)}}\Big) b_{121}^{(1)} \\ &= \sum_{j=1}^{d} (-1)^{d-j} \frac{A^{(1)}}{A^{(j)} A^{(j+1)}} [b_{220}^{(j)} A_0^{(j)} + b_{130}^{(j)} A_1^{(j)}] + (-1)^d b_{121}^{(1)}. \end{aligned}$$

This yields (38) if $d = 2N$ and (39) if $d = 2N + 1$. This completes the proof of the theorem. □

Remark In Theorem 6, if $\triangle_v$ is a quadrilateral with v being the intersection point of the two diagonal lines, the vertex v is called a singular vertex. In this case, equality (38) in Theorem 6 is an identity. This coincides with a known result.

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