# Chapter 5 probability

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# 5.1 PROBABILITY RULES

Some basic definition:

1. Probability---can be defined as the chance of an event occurring.
2. Probability experiment---- is a chance process that leads to well-defined results called outcomes.
3. Outcome---an outcome is the result of a single trial of a probability experiment.

Ex: Tossing a fair coin is a experiment, if I toss once, we call it one trial, toss twice, we call it 2 trials, and so on.

when I toss a coin once, there are two possible outcomes: head or tail.

# 5.1 PROBABILITY RULES

4. Sample space---denoted as “s”, is the collection of all possible outcomes.

5. Event--- an event is any collection of outcomes from a probability experiment. An event may consist one or more than one outcome. Usually , the event is denoted as capital E. If one event has one outcome, sometimes called simple events.

Ex:  a probability experiment consists of rolling a single fair die:

a) indentify the outcomes of the probability experiments.

b) determine the sample space.

c) determine the event E= roll an even number.

# 5.1 PROBABILITY RULES

Answer: a) the possible outcomes from rolling a single fair die are rolling a one={1}, rolling a two={2}, rolling a three={3}, rolling a four={4}, rolling a five={5} and rolling a six={6}.

b) All the possible outcomes forms the sample space,

S={1,2,3,4,5,6}

c) The event E="rolling a even number"={2, 4, 6}

**Probability model:** lists the possible outcomes of a probability experiment and each outcome's probability. A probability model must satisfy rules i) and ii).

**1. Rules of probabilities:**

i) the probability of any event E, P(E), must be greater than or equal to 0 and less than or equal to 1. That is,

$$0 \leq P(E) \leq 1$$

where P(E) means "the probability that event occurs."

ii) the sum of the probabilities of all outcomes must equal to 1. That is, if the sample space $S=\{e_1, e_2, \ldots, e_n\}$, then:

$$P(e_1)+p(e_2)+\ldots+p(e_n)=1$$

Ex: Toss a fair coin: (probability model): p(head)+p(tail)=1

| Outcomes | head | tail |
|---|---|---|
| Probability | 1/2 | 1/2 |

# 5.1 PROBABILITY RULES

iii) if an event E is certain, then the probability of E,

$$p(E)=1$$

Ex: A single die is rolled, what is the prob. of getting a number less than 7?

Because all the possible outcomes are less than 7, so this is a certain event, and the prob. of getting a number less than 7 is 1.

iv) If an event E is impossible, the probability of this event is 0.

Ex: Toss a fair coin, what is the prob.of getting a head and a tail at the same time?

Tossing a fair coin, you can only get a head or tail at one time, you cannot get both at the same time, so this event is impossible, and the prob. is zero.

v) If an event prob. close to zero, means this event occurs unlikely(also called unusual event); near to 50%,means a 50-50 chance to occur; close to 1, means this event is highly likely to occur.

**2. Empirical approach:**

The probability of an event E is approximately the number of times event E is observed divided by the number of repetitions of the experiment.

$$P(E) \approx \text{relative frequency of E} = \frac{\text{frequency of E}}{\text{number of trials in total}}$$

Note: empirical approach relies on actual experience to determine the likelihood of outcomes.

Ex: Hospital records indicated that maternity patients stayed in the hospital for the number of days shown in the below distribution. Calculate the probability model.(Hint: you should find out the total trials first, it is 127)

| Number of days stayed | frequency | probability |
|---|---|---|
| 3 | 15 | 15/127=0.12 |
| 4 | 32 | 32/127=0.25 |
| 5 | 56 | 0.44 |
| 6 | 19 | 0.15 |
| 7 | 5 | 0.04 |

# 5.1 PROBABILITY RULES

Questions: From the probability model, would you like to tell what the probability is for a patient staying exactly 5days, 7 days?

**<u>3. Classical method:</u>**

a) **Assume all outcomes in the sample space are equally likely to occur.**

b) **Equally likely outcomes means each outcome has the same probability of occurring.**

c) **Formula for classical method:**

$$\mathbf{P(E)=} \frac{\text{number of outcomes that E can occur}}{\text{number of all possible outcomes in sample space}} = \frac{N(E)}{N(S)}$$

**Ex:** A pair of fair dice is rolled.

a) compute the probability of rolling a seven.

b) compute the probability of rolling a two.

c) comment on the likelihood of rolling a seven versus rolling a two.

# 5.1 PROBABILITY RULES

Key: we need to count all the possible outcomes in the sample space and count the number of outcomes the event can occur.

In rolling a pair fair dice, there are 36 equally likely outcomes showing below:

| 1, 1 | 2,1 | 3,1 | 4,1 | 5,1 | 6,1 |
|---|---|---|---|---|---|
| 1,2 | 2,2 | 3,2 | 4,2 | 5,2 | 6,2 |
| 1,3 | 2,3 | 3,3 | 4,3 | 5,3 | 6,3 |
| 1,4 | 2,4 | 3,4 | 4,4 | 5,4 | 6,4 |
| 1,5 | 2,5 | 3,5 | 4,5 | 5,5 | 6,5 |
| 1,6 | 2,6 | 3,6 | 4,6 | 5,6 | 6,6 |

# 5.1 PROBABILITY RULES

Answer :

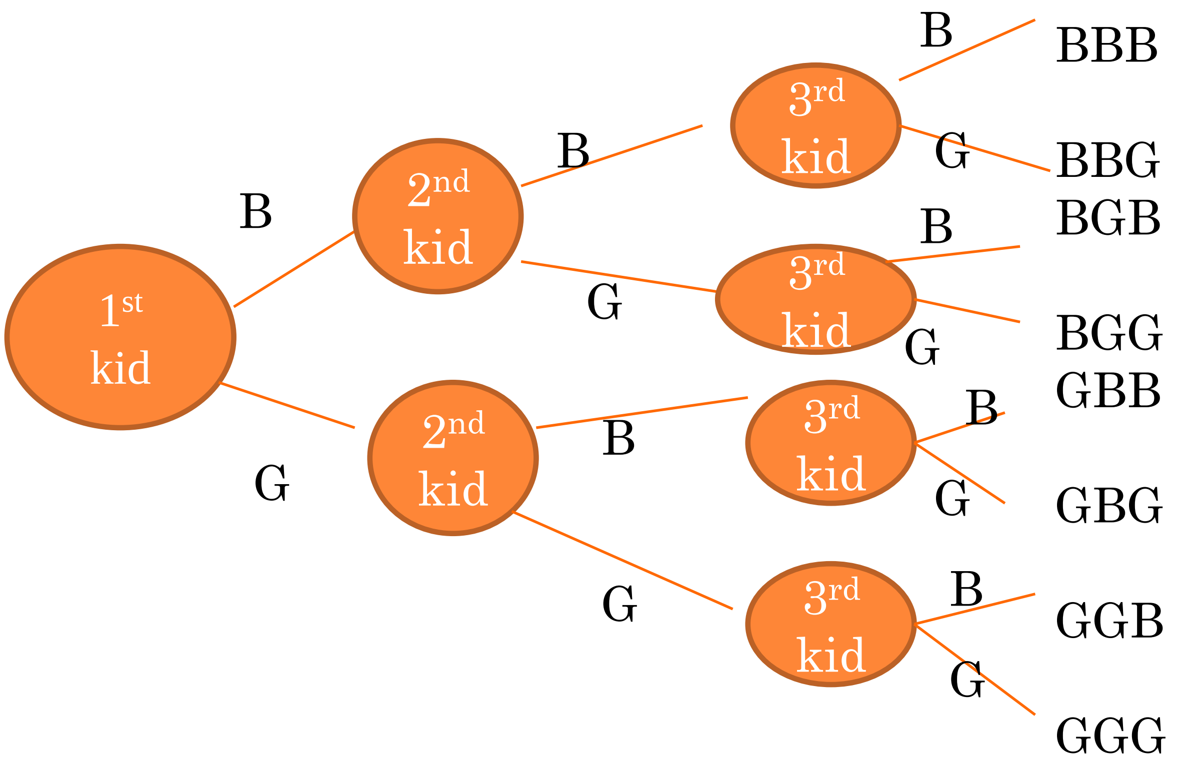
a) The event E="rolling a seven"={(1,6), (2,5),(3,4), (4,3), (5,2), (6,1)} has 6 outcomes. So N(S)=36, and N(E), and P(E)=N(E)/N(S)=6/36=1/6≈0.167
b) The event E="rolling a two"={(1,1)} has one outcome, and P(E)=N(E)/N(S)=1/36≈0.028
c) Because P(roll a seven)=6/36=1/6 and

   p(roll a two)=1/36, rolling a seven is six times as likely as rolling a two. In other words, in 36 rolls of the dice, we expect to observe about 6 sevens and only 1 two.

# 5.1 PROBABILITY RULES

**Tree diagram:** is a device consisting of line segments beginning a starting point and also from the outcome point.

**Ex:** find the sample space for the gender of the children if a family has 3 children. Using B for boy and G for girl.




The sample space is BBB,BBG,BGB, BGG,GBB,GBG, GGB and GGG.

# 5.1 PROBABILITY RULES

**Law of large numbers:**

As the numbers of trials increases, the empirical probability is approaching classical probability.

Ex: if I toss a coin, I expect the probability of getting a head is ½, if I toss 100 times, I will get exactly 50 times head? Not all the time, but as the number of toss increasing, the probability of getting a head will very close to ½.

Law of large numbers has a large application in insurance fields and other industries.

**Subjective probability:**

the probability is based on an educated guess or estimate.

Ex: I will say, 90% of students will pass this statistics course according to my teaching experience.

# 5.2 ADDITION RULE AND COMPLEMENTS

We think about the two events when a single fair die is rolled:

First situation: getting an odd number and getting an even number.

( This is definitely impossible to occur, it will occur either odd number or even, but will never occur both at the same time)

Second situation: getting a number of 4 and get a even number.

(this could happen at the same time)

1. **Disjoint events(mutually exclusive events):**

Two events are mutually exclusive events if they cannot occur at the same time( i.e., they have no outcomes in common).

addition rule for mutual exclusive events:

$$P(A \text{ or } B)=P(A)+P(B)$$

# 5.2 ADDITION RULE AND COMPLEMENTS

Ex: a day of the week is selected at random, find the probability that is a weekend day.

Answer: we know only Saturday and Sunday are weekend days.

We define the event A=Saturday; the event B=Sunday

The event A and B is mutually exclusive events, because they cannot occur at the same time, you cannot say today is Saturday and Sunday. So, the probability that is a weekend day is:

$$P(A \text{ or } B)=P(A)+P(B)=1/7+1/7=2/7$$

# 5.2 ADDITION RULE AND COMPLEMENTS

Ex: a single card is drawn from a deck, find the probability it is a king or a queen.

Answer: we define the event A=getting a king, B=getting a queen.

P(A)=4/52 P(B)=4/52

SO, P( A or B)=4/52+4/52=8/52=2/13

**Note:** if events A,B,C,D,… are mutually exclusive events, then we have the formula:

**P(A or B or C or D or…)=P(A)+P(B)+P(C)+P(D)+…**

# 5.2 ADDITION RULE AND COMPLEMENTS

2. General addition rule ( For any two events)

P(A or B)=P(A)+P(B)-P(A and B)

If two events are mutually exclusive, then P(A and B)=0

Ex: in a hospital, there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

Answer: P(nurse or male)==?

first to figure out sample space shown below:

| staff | male | female | total |
|---|---|---|---|
| nurse | 1 | 7 | 8 |
| physician | 2 | 3 | 5 |

Total=13

## 5.2 ADDITION RULE AND COMPLEMENTS

P(nurse or male)=p(nurse)+p(male)-P(nurse and male)

=8/13+3/13-1/13

= 10/13

Ex: the probability of getting a number 4 or getting an even number for rolling a die.

Answer: define the event A= getting a number 4

B=getting an even number

P(A or B)=P(A)+P(B)-P(A and B)

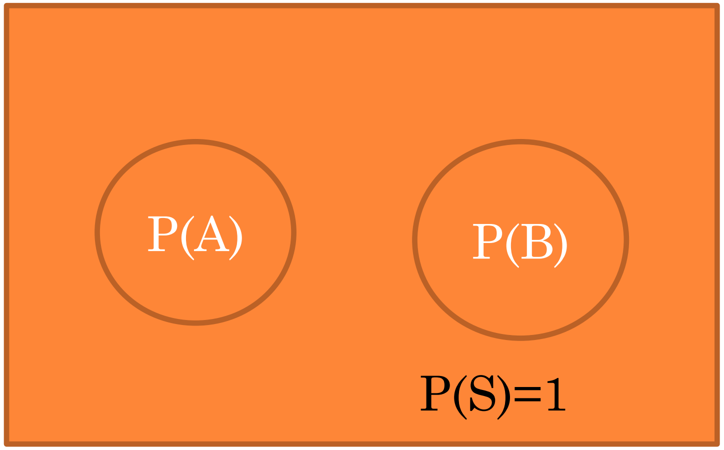
=1/6+3/6-1/6=1/2

**Note:** for any three events, we will have the formula:

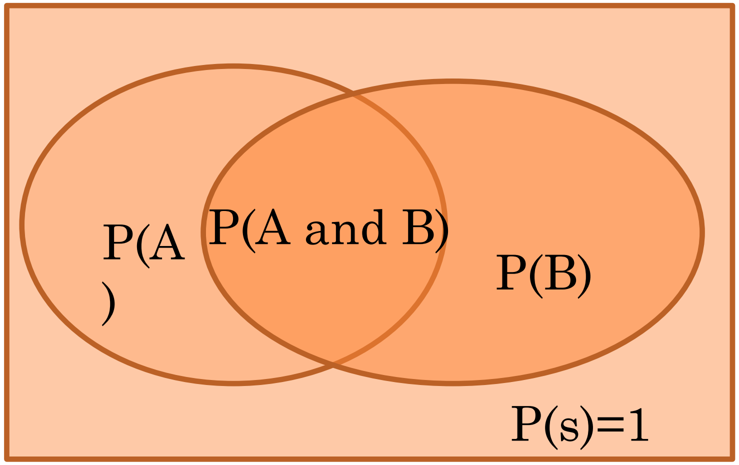
**P(A or B or C)=P(A)+P(B)+P(C)-P(A and B)-P(A and C)-P(B and C)**

# 5.2 ADDITION RULE AND COMPLEMENTS

Venn diagram to show different events:




mutually exclusive events




Non-mutually exclusive events

# 5.2 ADDITION RULE AND COMPLEMENTS

3. Complement of an event:

Let S denote the sample space of a probability experiment and let E denote an event. The complement of E, denoted $E^c$ , is all outcomes in the sample space S that are not outcomes in the event E.

Comments:

a) E and $E^c$ are mutually exclusive events.

b) $P(E \text{ or } E^c) = P(E) + P(E^c) = P(S) = 1$

Ex: if the probability of passing this course is 91%, what's the probability of failing this course?

Answer: P( passing this course)=91%, and the event of failing the course is the complement of passing the course, so P(failing the course)=1-91%=9%

# 5.3 INDEPENDENCE AND THE MULTIPLICATION RULE

**<u>1. Independent events & dependent events</u>**

If event A occurs does not affect the probability of event B occurring, the two events are independent events.

If event A occurs affect the probability of event B occurring, the two events are dependent events.

Ex: 1) first tossing a coin, and then tossing another coin, whether the second results will not be affected by the first results.

independent events

<u>Independent events vs. disjoint events:</u>

Disjoint events means if one event occurred, then another would not occur.

Independent events means one event occurs does not affect the probability of another event occurs.

# 5.3 Independence and the Multiplication Rule

<u>Formula for independent events:</u>

If event A and B are independent, then :

P(A and B)=P(A)*P(B)

Ex: A card is drawn from a regular deck and then put it back; a second card is drawn. Find the probability of getting a queen and then an ace.

answer: the event A "getting a queen in the first time" and the event B " getting an ace in the second time" are independent events.

P(A)=4/52 AND P(B)=4/52

P(A and B)=P(A)*P(B)=(4/52)*(4/52)=16/2704=1/169

# 5.3 INDEPENDENCE AND THE MULTIPLICATION RULE

**Multiplication rule for n independent events:**

If events A, B,C,… are independent, then

P(A and B and C and …)=P(A)*P(B)*P(C)…

Let's go over an example on page 252.

Note: there are some terms in probability calculation such as "at least" or "at most"

"at least" means greater than or equal to"

"at most" means less than or equal to"

Read Example 4 on page 253

# 5.4 CONDITIONAL PROBABILITY AND THE GENERAL MULTIPLICATION RULE

**1. Conditional probability:**

The notation $P(A \mid B)$ is read " the probability of event A given event B." It is the probability of event A occurs, given that event B has occurred.

Ex: if you roll a single fair die, what is the probability that the die comes up 3? Now suppose that the die is rolled a second time, but we are told the outcome will be an odd number. What is the probability that the die comes up 3?

Ans: In the first instance, there are six possibilities in the sample space, S={1,2,3,4,5,6}.so P(3)=1/6.

In the second instance, there are three possibilities in the sample space, because the only possible outcomes are odd, so S={1,3,5}, we have $P(3 \mid \text{outcome is odd})=1/3$, which is read " the probability of rolling a 3, given that the outcome is odd, is one-third".

# 5.4 CONDITIONAL PROBABILITY AND THE GENERAL MULTIPLICATION RULE

**Conditional probability rule:**

If A and B are any two events, then $P(A \mid B) = \dfrac{P(A \text{ and } B)}{p(B)} = \dfrac{N(A \text{ and } B)}{N(B)}$

The probability of event A occurring, given the occurrence of event B, is found (1) by dividing the probability of A and B by the probability of B, or (2) is found by dividing the number of outcomes in both A and B by the number of outcomes in B. These two methods in fact are the same.

Ex: a recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat, the results of the survey are shown.

| gender | yes | no | total |
|---|---|---|---|
| male | 32 | 18 | 50 |
| female | 8 | 42 | 50 |
| total | 40 | 60 | 100 |

# 5.4 CONDITIONAL PROBABILITY AND THE GENERAL MULTIPLICATION RULE

Q: 1)the response is yes, given that the response was a female.

2)the respondent was a male, given that the respondent answered no.

Ans: 1) P(yes | female)=N( yes and female)/ N(female)

=8/50=4/25

2)P(male | answer no)=N(male and answer no)/N(answer no)

=18/60=3/10

**2. General multiplication rule:**

the probability that two events A and B both occur is :

P( A and B)=P(A)*P(B | A) in fact, it is another form of conditional probability.

## 5.4 CONDITIONAL PROBABILITY AND THE GENERAL MULTIPLICATION RULE

Ex: A person owns a collection of 30 CDs, of which 5 are country music. If 2CDs are selected at random, find the probability that both are country music.

Ans: we define the event A as the first selected country CD, and B as the second selected country CD.

$$P(A \text{ and } B)=P(A)*P(B|A)$$

$$=(5/30)*(4/29)$$

$$=20/870$$

$$=2/87$$

# 5.4 CONDITIONAL PROBABILITY AND THE GENERAL MULTIPLICATION RULE

Note:

If event A and B are independent events, then we have:

a) $P(A|B)=P(A)$ or $P(B|A)=P(B)$

b) $P(A \text{ and } B)=P(A)*P(B)$

If event A, B,C,D,…are independent events, then we have:

$P(A \text{ and } B \text{ and } C \text{ and } D\ldots)$

$=P(A)*P(B)*P(C)*P(D)\ldots$

# 5.5COUNTING TECHNIQUES

**1. <u>Fundamental counting rule:</u>**

In a sequence of n events in which the first one has $K_1$ Possibilities, and the second one has $K_2$ possibilities, and the third one has $K_3$ possibilities and so forth; the total number of possibilities of the sequence will be : $(K_1) * (K_2) * (K_3)\ldots$

Ex: the digits 0,1,2,3 and 4 are used in a four-digit ID card, how many different cards are possible if repetition are permitted?

Ans: From the counting rule, there are 5*5*5*5=625 choices.

| | | | |
|---|---|---|---|
| 5 choices | 5 choices | 5 choices | 5 choices |



If without repetition, how many possible choices?

Ans: 5*4*3*2=120

# 5.5 COUNTING TECHNIQUES

Factorial formulas: (for $n \geq 0$ is an integer)

1) $n! = n(n-1)(n-2)\ldots3*2*1$

2) $0! = 1$

**2. permutations:**

a permutation is an arrangement of n objects in a specific order. The arrangement of n objects in a specific order using r objects at a time is called a permutation of n objects taking r objects at a time. ( repetition are not allowed). It is written as $nP_r$ , and the formula is :

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

# 5.5 COUNTING TECHNIQUES

Ex: there are 8 persons in a small group, how many different ways to choose 2 persons as director and associate director, respectively?

Ans:

$$ {}_8p_2 = \frac{8!}{(8-2)!} = (8!)/(6!) = 8*7 = 56 $$

**2. combinations:**

A selection of distinct objects without regard to order is called a combination. (repetition are not allowed). The number of combinations of r objects selected from n objects is like:

$$ {}_nC_r = \frac{n!}{(n-r)!r!} $$

# 5.5 COUNTING TECHNIQUES

Ex: in a club, there are 7 women and 5 men, A committee of 3 women and 2 men is to be chosen, how many different possibilities are there?

Ans: First, we must choose 3 women from 7 women, there are $_7C_3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!\,3!} = 35$ different choices;

second, we need choose 2 men from 5 men, there are

$_5C_2 = \frac{5!}{(5-2)!2!} = \frac{5!}{3!\,2!} = 10$ different choices.

third, by the fundamental counting rule , the total number of different ways is $\left({}_7C_3\right)\left({}_5C_2\right)$ =(35) (10)=350

# 5.5 COUNTING TECHNIQUES

Note : there is a relation between ${}_nC_r$ and ${}_nP_r$ :

$${}_nC_r = \frac{{}_nP_r}{r!}$$

Section 5.6 give some examples about the techniques we learned in chapter, you should read this section and understand each question.