Abstract: We offer students a modeling opportunity with logistic growth, harvesting, and diffusion in analyzing ocean fisheries which was adapted from [3, 4], in particular, that of the Atlantic cod. We help students build models, ever more complex, to capture physical realities. At each stage we ask students to reflect on the model and the implications.

Keywords: fishery, harvest, management, logistic growth, Atlantic cod

Tags: first order, differential equation, ordinary, partial, nonlinear, diffusion

STATEMENT

The logistic differential equation is often used to model the spread of disease as well as population growth. Can we use it to model sustainable harvesting of natural resources?

Fish are a valuable source of protein, and many people have lived to a large extent on fish and other seafood. Over fishing has driven many stocks of fish to near extinction [5, 6]. This applies in particular to the Mediterranean Sea, the Baltic Sea, and the North Atlantic Ocean. The collapse of the Newfoundland or Baltic sea cod fisheries should be taken as a severe warning that the fishery industry needs more careful controls [7]. With appropriate stock assessment data, mathematical models can be used to derive possible management strategies, which may aid the supervision of this industry.

U.S. stocks of Atlantic cod came close to commercial collapse in the mid-1990s. The 2012 assessments of Gulf of Maine and Georges Bank cod indicated both stocks are seriously over fished and are not recovering as quickly as expected. Based on these assessments, quotas for both stocks
were significantly reduced in 2013 to help ensure over fishing does not occur and that these stocks rebuild. The Gulf of Maine cod quota was cut by 80\%, and the Georges Bank cod quota was cut by 61\%. National Oceanic and Atmospheric Administration (NOAA) Fisheries and the New England Fishery Management Council continue to work on management measures that will further protect cod stocks and provide opportunities for fishermen to target other healthy fish stocks instead of cod [10]. Now NOAA asks you to model the fish stock with harvesting in St. Georges Bank in order to fish sustainably.

**Part I: Modeling Fish Harvesting**

Let us consider the cod existing in a continuous, finite region with logistic growth and carrying capacity $K$, where the harvesting of cod is proportional to the stock available.

1. Define your variables and write down the differential equation for the Atlantic cod with harvesting proportional to the stock available. Suppose $u(t)$ is the population of cod at time $t$, $r$ is the growth rate of the cod, $h$ is the constant harvesting rate, $u_0$ is the initial cod population. Then a reasonable model [2] is given by:

$$\frac{du}{dt} = ru \left(1 - \frac{u}{K}\right) - hu, \quad 0 < t < T,$$

$$u(0) = u_0,$$
where $T$ is the length of the fishing period.

2. Before solving the differential equation (1), what do you expect of the behavior of the fish stock? E.g., is the fish stock increasing or decreasing?

3. What kind of differential equation is this? Is it linear or nonlinear? What kind of method might you use to solve it?

<table>
<thead>
<tr>
<th>Year</th>
<th>$X_t$</th>
<th>$h_t$</th>
<th>Year</th>
<th>$X_t$</th>
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<td>0.350394</td>
<td>2009</td>
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Table 1. Annual (1978-2008) values of Atlantic cod biomass in metric tons, $X_t$, and harvest rate, $h(t)$, in metric tons in Georges Bank from [11].

4. Find the solution of the model. Atlantic cod biomass (in metric tons) and harvest rate in Georges Bank [11] are given in Table 1 from 1978 to 2008, could you estimate the parameters $r$ and $K$ from the data?

5. What is the long term behavior of the cod population? What does harvesting do for the cod? Can we conserve the cod population? Using the data in Table 1, if we increase the constant harvest rate, $h$, to be 0.4, how will the population of Atlantic Cod change over time?

6. Constant harvesting may not be realistic. What other harvesting strategy can you think of? Address the issues (1) - (5) above for this harvesting strategy.

Part II: Challenge: Farm Fishing

There are several recent issues of *Natural Resource Modeling Journal* devoted to fishery management and we call attention to the survey paper [9]. The author traces the development of fisheries models
from 1900 to the 21st century. He points out the fact that future fishery models will need to better address habitat and spatial concerns and to understand the effects of harvesting on the ecosystem. Managers have begun to use spatial management instruments, in the form of both permanent and temporary closures.

Assuming we implement fish farming for cod, so that cod live in a bounded region, for simplicity, we assume one-dimensional spatial domain with logistic growth and continuous diffusion [1]. We assume zero stock at the ends of the habitat, depicting a situation in which the habitat everywhere outside the domain in question cannot support the resource.

Let’s think about how to add the spatial component. Hint to students: the term $u_{xx}$ can describe the diffusion [1]. In the Appendix, you may find an intuitive way to derive and understand the diffusion partial differential equation. Note the students from Differential Equations II will be comfortable using $u_{xx}$, if this part is too challenging, you may skip questions 7-9.

7. We write down the new differential equation that includes the spatial component, let $u = u(x, t)$.

$$u_t = D u_{xx} + r u \left(1 - \frac{u}{K}\right) - h u, \quad 0 < x < l, \quad 0 < t < T \quad (2)$$

What kind of initial and/or boundary conditions do we need to solve this problem? Hint: We need to specify both initial condition $u(x, 0) = f(x)$ and boundary conditions $u(0, t) = g_1(t)$ and $u(l, t) = g_2(t)$ (in our case, we have $g_1(t) = g_2(t) = 0$ since we have hostile environment). Interpret these conditions in this fishery setting. $D$ is the diffusion rate.

8. How might a solution strategy differ from Part I used in solving (1)? What do you anticipate from your model by adding the spatial component?

9. Ponder: For an even more realistic situation, suppose the harvesting $h$ depends on the spatial location and the time. Write out a differential equation which might model this phenomenon.

What is the difference between this newer result and your result from Question 8?

Acknowledgement

We used a simplified version of the models from W. Ding, G.E. Herrera, H.R. Joshi, S. Lenhart and M.G. Neubert [3, 4].

COMMENTS

Students will feel comfortable solving Part I and be able to answer most questions from (1) - (7). For Part II, our original idea was a motivation for the introduction to partial differential equations for Differential Equations II class and for a graduate Mathematical Modeling class for Computational Science Program. The object is to give the students an opportunity to see more realistic models on fishery harvesting by adding the spatial component. Some students already know the diffusion term $u_{xx}$, if the students do not know this, it would be easier to tell them directly to use second order derivative for the diffusion as [1] suggests and not to spend too much time discussing it.
Appendix: Diffusion

Description of movement arising from many short movements in random directions. Suppose an organism moves along a line: moving a distance $\Delta x$ to the left with probability $\frac{1}{2}$, moving a distance $\Delta x$ to the right with probability $\frac{1}{2}$. Suppose $p(x, t) =$ probability that the organism is at location $x$ at time $t$, then

$$p(x, t) = \frac{1}{2}p(x + \Delta x, t - \Delta t) + \frac{1}{2}p(x - \Delta x, t - \Delta t). \quad (3)$$

Subtracting $p(x, t - \Delta t)$ from both sides and divided by $\Delta t$, we have

$$\frac{p(x, t) - p(x, t - \Delta t)}{\Delta t} = \frac{p(x + \Delta x, t - \Delta t) - 2p(x, t - \Delta t) + p(x - \Delta x, t - \Delta t)}{2\Delta t}. \quad (4)$$

Imposing the scaling $\frac{(\Delta x)^2}{\Delta t} = 2D$, we obtain

$$\frac{p(x, t) - p(x, t - \Delta t)}{\Delta t} = \frac{D\left[p(x + \Delta x, t - \Delta t) - 2p(x, t - \Delta t) + p(x - \Delta x, t - \Delta t)\right]}{(\Delta x)^2}. \quad (5)$$

As $\Delta x, \Delta t$ go to 0, $\frac{p(x, t) - p(x, t - \Delta t)}{\Delta t}$ goes to $p_t(x, t)$, and

$$\frac{\left[p(x + \Delta x, t - \Delta t) - 2p(x, t - \Delta t) + p(x - \Delta x, t - \Delta t)\right]}{(\Delta x)^2}$$

goes to $p_{xx}(x, t)$. So we have the diffusion equation

$$p_t = Dp_{xx}.$$

REFERENCES


