Optimal Harvesting of a Spatially Explicit Fishery Model

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1. Motivation: Benefits of marine reserves?

Neubert (Ecology Letters, 2003) studied the fishery management problem:
Maximize the yield
\[ J(E) = \int_0^L q(x) E(x) \, dx \]
subject to
\[ -\frac{d^2 E(x)}{dx^2} + r \left( 1 - \frac{E(x)}{K} \right) - q(x) E(x) \leq 0, \quad 0 < x < L, \]
\[ N(0) = N(L) = 0. \]

2. Neubert's Results

- No-take marine reserves are always part of an optimal harvest strategy to maximize yield.
- The sizes and locations of the optimal reserves depend on a dimensionless length parameter.
- For small values of this parameter, the maximum yield is obtained by placing a large reserve in the center of the habitat.
- For large values of this parameter, the optimal harvesting strategy is a spatially 'chatting' control with infinite sequences of reserves alternating with intervals of intense fishing.

3. Our Fishery Model: Steady-State

\[ -\Delta u = ru(1 - u) - h(x)u, \quad x \in \Omega, \]
\[ u = 0, \quad x \in \partial \Omega. \]

where \( h(x) \) is the fish density, \( r \) is the growth rate, \( b(x) \) is the harvesting depending on the location of fish, \( \Omega \in \mathbb{R}^p \), smooth and bounded domain. Note that it is a solution. But we seek solutions that are positive in \( \Omega \).

4. Two Optimal Control Problems

Control Set I:
\[ U_1 = \{ h(x) \in L^2(\Omega) \mid 0 \leq h(x) \leq h_{\text{max}} \ \text{a.e.} \}. \]

Goal I: Maximizing the yield and minimizing the cost of fishing
\[ J_1(h) = \int_\Omega h(x)u(x) \, dx - \int_\Omega (B_1 + B_2 h(x)) \, dx, \quad h \in U_1. \]

Control Set II:
\[ U_2 = \{ h(x) \in L^2(\Omega) \mid 0 \leq h(x) \leq h_{\text{max}} \ \text{a.e.} \}. \]

Goal II: Maximizing the yield and minimizing the variation of the fishing effort
\[ J_2(h) = \int_\Omega h(x)u(x) \, dx - \int_\Omega \nabla h^2 \, dx, \quad h \in U_2. \]

5. Optimality System I

- State equation
\[ -\Delta u = ru(1 - u) - h(x)u, \quad x \in \Omega, \]
\[ u = 0, \quad x \in \partial \Omega. \]

- Adjoint equation
\[ -\Delta p = r(1 - 2u)p + hp, \quad x \in \Omega, \]
\[ p = 0, \quad x \in \partial \Omega. \]

- Characterization of optimal control
\[ h^*(x) = \min \{ \max \{ 0, \frac{u - pu - B_2}{2B_2} \} h_{\text{max}} \}. \]

6. Numerical Examples for \( J_1 \): 1-D case, \( B \) effect

Set \( B = 0.1, 0.5, 1.25, 2, 5, 10 \)

7. Numerical Examples for \( J_1 \): 1-D case, small \( B \) effect

Set \( B = 0.1, 0.05, 0.01 \)

8. Numerical Examples for \( J_2 \): 2-D case, \( B \) effect

9. Numerical Examples for \( J_2 \): 2-D case, domain size effect: \( (0.5) \times (0.3) \) vs \( (0.25) \times (0.25) \)

10. Numerical Examples for \( J_2 \): 2-D case, small \( B \) values

Set \( B_1 = 0, B_2 = 0.83 \)

11. Maximizing the yield with No-flux Boundary Condition

If \( B_2 = B_2 = 0 \) in \( J_2(h) \), and we have Neumann (no-flux) boundary condition, then the optimal control and optimal state are
\[ h^*(x) = \frac{1}{2}, \quad u^*(x) = \frac{1}{2}. \]

12. Generalize Neubert's Results

Maximizing the yield in Multidimension
If \( B_1 = B_2 = 0 \) in \( J_2(h) \), then the optimal control is given by
\[ h^*(x) = \begin{cases} 0, & p > 1; \\ h_{\text{max}}, & p < 1; \\ \frac{r}{2}, & p = 1. \end{cases} \]

13. Optimality System II

- State equation
\[ -\Delta u = ru(1 - u) - h(x)u, \quad x \in \Omega, \]
\[ u = 0, \quad x \in \partial \Omega. \]

- Adjoint equation
\[ -\Delta p = r(1 - 2u)p + hp, \quad x \in \Omega, \]
\[ p = 0, \quad x \in \partial \Omega. \]

- Characterization of optimal control
\[ \min \{ \max \{ 0, \frac{u - pu - 2A h}{2A h} \} h_{\text{max}}, 0 \} = 0. \]

14. Numerical Examples for \( J_2 \): small production

15. Numerical Examples for \( J_2 \) vary \( A \): 1, 2.5, 5, 10

16. Conclusion

- If we want to maximize yield and minimize cost, then increasing the cost coefficients \( B_1 \) or \( B_2 \), will decrease optimal harvesting.
- With small \( B_1 \) and \( B_2 \) the harvest control is concentrated near the boundary.
- If we only want to maximize yield, then reserve is part of the optimal harvesting strategy.
- The problem of maximizing yield only with Neumann boundary condition gives a simple optimal control, a singular case.
- For \( J_1 \), the optimal benefit increases when domain size increases.
- If we want to maximize yield and minimize variation in fishing effort, then increasing the variation coefficient \( A \) will reduce optimal harvesting.